ON THE ORDER OF THE SCHUR MULTIPLIER OF A PAIR OF FINITE p-GROUPS II

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Abstract. Let $G$ be a finite $p$-group and $N$ be a normal subgroup of $G$ with $|N| = p^n$ and $|G/N| = p^m$. A result of Ellis (1998) shows that the order of the Schur multiplier of such a pair $(G, N)$ of finite $p$-groups is bounded by $p^{\frac{1}{2}n(2m+n-1)}$ and hence it is equal to $p^{\frac{1}{2}n(2m+n-1)-t}$ for some non-negative integer $t$. Recently, the authors have characterized the structure of $(G, N)$ when $N$ has a complement in $G$ and $t \leq 3$. This paper is devoted to classification of pairs $(G, N)$ when $N$ has a normal complement in $G$ and $t = 4, 5$.

1. Introduction

By a pair of groups $(G, N)$ we mean a group $G$ with a normal subgroup $N$. In 1998, Ellis [2] defined the Schur multiplier of a pair $(G, N)$ to be the abelian group $M(G, N)$ appearing in a natural exact sequence

$$
H_3(G) \rightarrow H_3(G/N) \rightarrow M(G, N) \rightarrow M(G) \rightarrow M(G/N) \rightarrow 0
$$

in which $H_3(G)$ is the third homology of $G$ with integer coefficients. He [2] also noted that for any pair $(G, N)$ of groups,

$$
M(G, N) \cong \ker(N \wedge G \rightarrow G),
$$

where $N \wedge G$ is the exterior product of $N$ and $G$. In particular, if $N = G$, then $M(G, G)$ is the usual Schur multiplier of $G$. 


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In 1956, Green [4] showed that if $G$ is a group of order $p^n$, then its Schur multiplier is of order at most $p^{\frac{n(n-1)}{2}}$ and hence equals to $p^{\frac{n(n-1)}{2}-t}$ for some non-negative integer $t$. Berkovich [1], Zhou [11], Ellis [3] and Niroomand [7, 8] determined the structure of $G$ for $t = 0, 1, 2, 3, 4, 5$ by different methods.

In 1998, Ellis [2] gave an upper bound for the order of the Schur multiplier of a pair of finite $p$-groups. He proved that if $G$ is a finite $p$-group with a normal subgroup $N$ of order $p^n$ and its quotient $G/N$ of order $p^m$, then the Schur multiplier of $(G, N)$ is bounded by $p^{\frac{1}{2}n(2m+n-1)}$ and hence equals to $p^{\frac{1}{2}n(2m+n-1)-t}$ for some non-negative integer $t$.

Let $(G, N)$ be a pair of groups and $K$ be the complement of $N$ in $G$. In 2004, Salemkar, Moghaddam and Saeedi [10] characterized the structure of such a pair $(G, N)$ when $t = 0, 1$ under some conditions. Recently, the authors [5] determined the structure of the pair $(G, N)$, for $t = 0, 1$ without any condition and also gave the structure of $(G, N)$ for $t = 2, 3$ when $K$ is normal. In this paper, we are going to determine the structure of $(G, N)$ for $t = 4, 5$ when $K$ is a normal subgroup of $G$.

In this paper, $D$ and $Q$ denote the dihedral and the quaternion group of order 8, $D_{16}$ denotes the dihedral group of order 16 and, $E_1$ and $E_2$ denote the extra special $p$-groups of order $p^3$ of odd exponent $p$ and $p^2$, respectively. Also $E_4$ denotes the unique central product of a cyclic group of order $p^2$ and a non-abelian group of order $p^3$, and $\mathbb{Z}_n^{(m)}$ denotes the direct product of $m$ copies of $\mathbb{Z}_n$.

The following result is essential to prove the main theorems.

**Theorem 1.1.** [2] Let $(G, N)$ be a pair of groups and $K$ be the complement of $N$ in $G$. Then

$$M(G) \cong M(G, N) \times M(K).$$

In 1907, Schur [6] gave an structure for the Schur multiplier of a direct product of finite groups. He showed that

$$M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1^{ab} \otimes G_2^{ab}).$$

As a consequence of this fact we have the following important result.

**Corollary 1.2.** Let $(G, N)$ be a pair of groups and $K$ be the complement of $N$ in $G$. Then

$$|M(G, N)| = |M(N)||N^{ab} \otimes K^{ab}|.$$

The following theorems give the structure of a finite $p$-group in terms of the order of its Schur multiplier.

**Theorem 1.3.** [3] Let $G$ be a group of prime-power order $p^n$ with $|M(G)| = p^{\frac{1}{2}n(n-1)-t}$. Then

i) $t = 0$ if and only if $G$ is elementary abelian;

ii) $t = 1$ if and only if $G \cong \mathbb{Z}_{p^2}$ or $G \cong E_1$;

iii) $t = 2$ if and only if $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $G \cong D$ or $G \cong \mathbb{Z}_p \times E_1$;

iv) $t = 3$ if and only if $G \cong \mathbb{Z}_{p^3}$, $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $G \cong Q$, $G \cong E_2$, $G \cong D \times \mathbb{Z}_2$ or $G \cong E_1 \times \mathbb{Z}_p \times \mathbb{Z}_p$.  


Theorem 1.4. Let $G$ be an abelian group of order $p^n$ with $|M(G)| = p^{\frac{1}{2}n(n-1)-4}$. Then $G$ is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(3)}$.

The following result can be easily obtained by using a method similar to the proof of Theorem 1.4:

Theorem 1.5. Let $G$ be an abelian group of order $p^n$ with $|M(G)| = p^{\frac{1}{2}n(n-1)-5}$. Then $G$ is isomorphic to $\mathbb{Z}_{p^3} \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(4)}$.

Theorem 1.6. Let $G$ be a non-abelian group of order $p^n$ with $|M(G)| = p^{\frac{1}{2}n(n-1)-4}$. Then $G$ is isomorphic to one of the following groups.

For $p = 2$,
1) $D \times \mathbb{Z}_2^{(2)}$;
2) $Q \times \mathbb{Z}_2$;
3) $\langle a, b \rangle a^4 = b^6 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2$;
4) $\langle a, b, c \rangle a^2 = b^2 = c^2 = 1, abc = bca = cab$;

For $p \neq 2$
5) $E_4$;
6) $E_1 \times \mathbb{Z}_p^{(3)}$;
7) $\mathbb{Z}_p^{(4)} \rtimes \theta \mathbb{Z}_p$;
8) $E_2 \times \mathbb{Z}_p$;
9) $\langle a, b \rangle a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = 1$;
10) $\langle a, b \rangle a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6; [a, b, b, b] = 1$;
11) $\langle a, b \rangle a^{p^2} = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1$ (if $p \neq 3$);

Theorem 1.7. Let $G$ be a non-abelian group of order $p^n$ with $|M(G)| = p^{\frac{1}{2}n(n-1)-5}$. Then $G$ is isomorphic to one of the following groups.
1) $D \times \mathbb{Z}_2^{(3)}$;
2) $E_1 \times \mathbb{Z}_p^{(4)}$;
3) $E_2 \times \mathbb{Z}_p^{(2)}$;
4) $E_4 \times \mathbb{Z}_p$;
5) extra special $p$-group of order $p^5$;
6) $\langle a, b \rangle a^{p^2} = b^2 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p$;
7) $\langle a, b \rangle a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1$;
8) $\langle a, b \rangle a^{p^2} = b^p = 1, [a, b, a] = [a, b, b, b] = 1, [a, b, b] = a^{np}$
   where $n$ is a fixed quadratic non-residue of $p$ and $p \neq 3$;
9) $\langle a, b \rangle a^{p^2} = 1, b^3 = 1, [a, b, a] = [a, b, b, b] = 1, [a, b, b] = a^6$;
10) $\langle a, b \rangle a^{p^2} = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, b] = [a, b, b, a] = 1$;
11) $D_{16}$;
12) $\langle a, b \rangle a^4 = b^4 = 1, a^{-1}ba = b^{-1}$;
13) $Q \times \mathbb{Z}_2^{(2)}$;
14) $(D \times \mathbb{Z}_2) \triangleright \triangleleft \mathbb{Z}_2$;
15) $(Q \times \mathbb{Z}_2) \triangleright \triangleleft \mathbb{Z}_2$;
16) $\mathbb{Z}_2 \times \langle a, b, c|a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$;

2. Main Results

In this section, let $(G, N)$ be a pair of groups such that $G \cong N \times K$ with $|N| = p^n$ and $|K| = p^m$. As mentioned before, Ellis [2] showed that $|M(G, N)| = p^{2n(2m+n-1)}$ for some non-negative integer $t$. Recently, all these pairs of finite $p$-groups are listed in [5] by the authors, when $t = 0, 1, 2, 3$. The aim of this paper is to characterize the structure of such pairs of finite $p$-groups, when $t = 4, 5$.

**Theorem 2.1.** By the above assumption, $t = 4$ if and only if $G$ is isomorphic to one of the following groups.

1) $G \cong N \times K$ where $N \cong \mathbb{Z}_p$ and $K$ is any group with $d(K) = m - 4$;
2) $G \cong N \times K$ where $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $K$ is any group with $d(K) = m - 2$;
3) $G \cong N \times K$ where $N \cong \mathbb{Z}_p^{(4)}$ and $K$ is any group with $d(K) = m - 1$;
4) $G = N \cong D \times \mathbb{Z}_2^{(2)}$;
5) $G = N \cong Q \times \mathbb{Z}_2$;
6) $G = N \cong \langle a, b|a^4 = b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2 \rangle$;
7) $G = N \cong \langle a, b, c|a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$;
8) $G = N \cong E_4$;
9) $G = N \cong E_1 \times \mathbb{Z}_p^{(3)}$;
10) $G = N \cong \mathbb{Z}_p^{(4)} \triangleright \triangleleft \mathbb{Z}_p$;
11) $G = N \cong E_2 \times \mathbb{Z}_p$;
12) $G = N \cong \langle a, b|a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$;
13) $G = N \cong \langle a, b|a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle$;
14) $G = N \cong \langle a, b|a^p = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b] = 1 \rangle$

($p \neq 3$);
15) $G = N \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^{(2)}$;
16) $G = N \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^{(3)}$;
17) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^{(2)}$;
18) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong Q$;
19) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong E_2$;
20) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong D \times \mathbb{Z}_2$;
21) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong E_1 \times \mathbb{Z}_p^{(2)}$;
22) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p$;
23) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong D$;
24) $G \cong N \times K$ where $K = \mathbb{Z}_p^2$ and $N \cong E_1 \times \mathbb{Z}_p$;
25) $G \cong N \times K$ where $K = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $N \cong \mathbb{Z}_{p^2}$;
26) $G \cong N \times K$ where $K = \mathbb{Z}_p^3$ and $N \cong E_1$;
27) $G \cong N \times K$ where $K = \mathbb{Z}_p^3$ and $N \cong \mathbb{Z}_{p^2}$.

**Proof.** The necessity of theorem follows from the fact that $G = N \times K$ and Corollary 1.2. For sufficiency, first suppose that $N$ is an elementary abelian $p$-group. Then $nm - 4 = nd(K)$ and using Corollary 1.2 we have $|N \otimes K^{ab}| = p^{nm-4} = p^{nd(K)}$. Hence $n(m - d(K)) = 4$ which implies that $n = 1, 2$ or 4. Therefore $N \cong \mathbb{Z}_p$ and $K$ is any group with $d(K) = m - 4$ or $N \cong \mathbb{Z}_p^{(2)}$ and $K$ is any group with $d(K) = m - 2$, or $N \cong \mathbb{Z}_p^{(4)}$ and $K$ is any group with $d(K) = m - 1$.

Now suppose that $N$ is not an elementary abelian $p$-group. Then using Corollary 1.2 we have $|N^{ab} \otimes K^{ab}| > p^{nm-4}$ and so $md(N) > nm - 4$ which implies that $m(n - d(N)) < 4$. Therefore $m = 0, 1, 2, 3$.

If $m = 0$, then $K = 1$ and $N$ is one of the groups which are listed in Theorems 1.4 and 1.6.

If $m = 1$, then $K = \mathbb{Z}_p$ and $d(N) = n - 1, n - 2$ or $n - 3$. It follows that $|N^{ab} \otimes K| = p^{n-1}, p^{n-2}$ or $p^{n-3}$ and so Corollary 1.2 implies that $|M(N)| = p^{\frac{d}{2}n^n-2}$, $p^{\frac{d}{2}n^n-2}$, or $p^{\frac{n^2-n-1}{2}}$, respectively. In the first case $N$ is $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p^2$, $\mathbb{Q}, E_2$, $D \times \mathbb{Z}_2$ or $E_1 \times \mathbb{Z}_p \times \mathbb{Z}_p$ and the other cases are impossible by Theorem 1.3.

If $m = 2$, then $d(N) = n - 1$ and $K = \mathbb{Z}_p \times \mathbb{Z}_p$ or $K = \mathbb{Z}_p^{(2)}$. In the first case $|N^{ab} \otimes K| = p^{2(n-1)}$ and so $|M(N)| = p^{\frac{d}{2}n^n-2}$. Therefore $N$ is $\mathbb{Z}_p \times \mathbb{Z}_p^2$, $D$ or $\mathbb{Z}_p \times E_1$. In the second case $N^{ab} \cong \mathbb{Z}_p^{(n-1)}$ or $N^{ab} \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^{(n-2)}$. If $N^{ab}$ is an elementary abelian $p$-group, then $|N^{ab} \otimes K| = p^{n-1}$ and so $|M(N)| = p^{\frac{n^2-n-8}{2}}$ which is impossible. If $N^{ab} \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^{(n-2)}$, then $|M(N)| = p^{\frac{n^2-n-8}{2}}$ which is impossible too.

If $m = 3$, then $d(N) = n - 1$ and $K$ is an abelian $p$-group of order $p^3$ or an extra special $p$-group of order $p^3$. In the first case we have three possibilities for $K$. The first possibility is $K \cong \mathbb{Z}_p^{(3)}$, and similar to the previous part, one can see that $|M(N)| = p^{\frac{d}{2}n^n-1}$ and so $N \cong E_1$ or $\mathbb{Z}_p^{(2)}$. The second possibility is $K \cong \mathbb{Z}_p^{(3)}$. This implies that $n = 1$ which is a contradiction. The third possibility is $K \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p$ which implies that $n = 2$ and $N \cong \mathbb{Z}_p^{(2)}$.

In the second case, if $K$ is an extra special $p$-group of order $p^3$, then $N^{ab} \cong \mathbb{Z}_p^{(n-1)}$ or $N^{ab} \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^{(n-2)}$. This implies that $|N^{ab} \otimes K| = |N^{ab} \otimes \mathbb{Z}_p^{(2)}| = p^{2n-2}$ and so $n = 1$ which is a contradiction. Hence the proof is complete.

**Theorem 2.2.** By the previous assumption, $t = 5$ if and only if $G$ is isomorphic to one of the following groups.

1) $G \cong N \times K$ where $N \cong \mathbb{Z}_p$ and $K$ is any group with $d(K) = m - 5$;
2) $G \cong N \times K$ where $N \cong \mathbb{Z}_p^{(5)}$ and $K$ is any group with $d(K) = m - 1$;
3) $G = N \cong D \times \mathbb{Z}_p^{(3)}$;
4) $G = N \cong E_1 \times \mathbb{Z}_p^{(4)}$;
5) $G = N \cong E_2 \times \mathbb{Z}_p^{(2)}$.
6) $G = N \cong E_4 \times \mathbb{Z}_p$;
7) $G = N \cong$ an extra special $p$-group of order $p^5$;
8) $G = N \cong \langle a, b| a^p = b^p = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle$;
9) $G = N \cong \langle a, b| a^p = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1 \rangle$;
10) $G = N \cong \langle a, b| a^p = b^p = 1, [a, b, a] = [a, b, b, b] = 1, [a, b, b] = a^{np},$
where $n$ is a fixed quadratic non-residue of $p$ and $p \neq 3$;
11) $G = N \cong \langle a, b| a^3, b^p = [a, b, a] = [a, b, b, b] = 1, [a, b] = a^6 \rangle$;
12) $G = N \cong \langle a, b| a^p = 1, b^p = [a, b, b, b, b] = 1, [a, b, b, b] = [a, b, a] = 1 \rangle$;
13) $G = N \cong D_{16}$;
14) $G = N \cong \langle a, b| a^4 = b^4 = 1, a^{-1}ba = b^{-1} \rangle$;
15) $G = N \cong Q \times \mathbb{Z}_2^{(2)}$;
16) $G = N \cong (D \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$;
17) $G = N \cong (Q \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$;
18) $G = N \cong \mathbb{Z}_2 \times \langle a, b, c| a^2 = b^2 = c^2 = 1, abc = bca = cab >$;
19) $G = N \cong \mathbb{Z}_p^3 \times \mathbb{Z}_p$;
20) $G = N \cong \mathbb{Z}_p^2 \times \mathbb{Z}_p^{(4)}$;
21) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong D \times \mathbb{Z}_2^{(2)}$;
22) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong Q \times \mathbb{Z}_2$;
23) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong \langle a, b, c| a^2 = b^2 = c^2 = 1, abc = bca = cab >$;
24) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong E_4$;
25) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong E_1 \times \mathbb{Z}_p^{(3)}$;
26) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$;
27) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong E_2 \times \mathbb{Z}_p$;
28) $G \cong N \times K$ where $K = \mathbb{Z}_p$ and $N \cong E_2 \times \mathbb{Z}_p$;
29) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong E_1 \times \mathbb{Z}_p^{(2)}$;
30) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong \mathbb{Z}_p^{(2)} \times \mathbb{Z}_p^2$;
31) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong Q$;
32) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong E_2$;
33) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(2)}$ and $N \cong D \times \mathbb{Z}_2$;
34) $G \cong N \times K$ where $K = \mathbb{Z}_p^2$ and $N \cong E_1$;
35) $G \cong N \times K$ where $K = \mathbb{Z}_p^3$ and $N \cong \mathbb{Z}_p \times \mathbb{Z}_p^2$;
36) $G \cong N \times K$ where $K = \mathbb{Z}_p^3$ and $N \cong \mathbb{Z}_p$;
37) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(3)}$ and $N \cong D$;
38) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(3)}$ and $N \cong E_1 \times \mathbb{Z}_p$;
39) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(3)}$ and $N \cong \mathbb{Z}_p \times \mathbb{Z}_p^2$;
40) $G \cong N \times K$ where $K$ is an extra special $P$-group and $N \cong \mathbb{Z}_p^2$;
41) $G \cong N \times K$ where $K = \mathbb{Z}_p^{(4)}$ and $N \cong E_1$;
42) $G \cong N \times K$ where $K = Z_p^{(2)} \times Z_p$ and $N \cong Z_{p^2}$.

**Proof.** The proof of this theorem is similar to the proof of the previous theorem so we left the details to the reader. Necessity is straightforward. For sufficiency, first suppose that $N$ is an elementary abelian $p$-group. Then $n(m - d(K)) = 5$, so $n = 1$ or 5. If $n = 1$, then $N \cong Z_p$ and $K$ is any group with $d(K) = m - 5$ and $n = 5$ which implies that $N \cong Z_p^{(5)}$ and $K$ is any group with $d(K) = m - 1$.

Suppose that $N$ is not an elementary abelian $p$-group. Then we have $|N^{ab} \otimes K^{ab}| > p^{m-5}$ by Corollary 1.2. It follows that $md(N) > nm - 5$ and thus $m(n - d(N)) < 5$ which implies that $m = 0, 1, 2, 3$ or 4. If $m = 0$, then $K = 1$ and $N$ is one of the groups that are listed in Theorems 1.5 and 1.7.

If $m = 1$, then $K = Z_p$ and $d(N) = n - i$, for $1 \leq i \leq 4$. Therefore by Corollary 1.2 $|M(N)| = p^{n(n-1)-(5-i)}$, for $1 \leq i \leq 4$, respectively. It follows that $N \cong Z_p^2 \times Z_p^{(3)}, D \times Z_p^{(2)}, Q \times Z_2, E_1, E_2 \times Z_p, N \cong Z_p^3$ or $\langle a, b, c|a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$ by Theorems 1.6, 1.4 and 1.3.

If $m = 2$, then $K = Z_p^2$ or $K = Z_p \times Z_p$ and $d(N) = n - 1$ or $d(N) = n - 2$. First suppose that $K = Z_p \times Z_p$. If $d(N) = n - 1$, then $|M(N)| = p^{n \frac{n+3}{2}}$. It follows that $N \cong Z_p \times Z_p \times Z_{p^2}, N \cong E_1 \times Z_p \times Z_p, N \cong Z_p, N \cong E_2$ or $N \cong D \times Z_2$. If $d(N) = n - 2$, then $|N \otimes K| = p^{2(n-2)}$. This implies that $|M(N)| = p^{n \frac{n+3}{2}}$ and so $n < 1$ which is impossible.

Now suppose that $K = Z_p^2$. If $d(N) = n - 1$ and $N^{ab} \cong Z_{p}^{(n-1)}$, then $|M(N)| = p^{n \frac{n+4}{2}}$ which implies that $n = 3$ and $|M(N)| = p^2$. Therefore $N \cong E_1$. If $d(N) = n - 1$ and $N^{ab} \cong Z_{p^2} \times Z_{p^{n-2}}$, then $n = 3$ or $n = 4$. For $n = 3$ there is not any structure for $N$ and $n = 4$ implies that $N \cong Z_{p^2} \times Z_p$. If $d(N) = n - 2$, then $N^{ab} \cong Z_{p}^{(n-2)}$ or $N^{ab} \cong Z_{p}^{3} \times Z_{p}^{n-3}$ or $N^{ab} \cong Z_{p}^{2} \times Z_{p}^{2} \times Z_{p}^{(n-4)}$. Therefore similar to the previous case one can see that $n < 5$ which is impossible.

If $m = 3$, then $d(N) = n - 1$ and $K$ is an abelian $p$-group of order $p^3$ or is an extra special $p$-group of order $p^3$. In the first case we have three possibilities for $K$. The first possibility is $K \cong Z_{p^3}$. If $N^{ab} \cong Z_{p}^{(n-1)}$, then $n = 1$ which is impossible and if $N^{ab} \cong Z_{p^2} \times Z_{p}^{n-2}$, then $N \cong Z_{p^2}$.

The second possibility is $K \cong Z_{p^2} \times Z_{p^2}$. In this case, there is no structure for $N$. The third possibility is $K \cong Z_{p}^{3}$. Thus $|M(N)| = p^{n \frac{n+4}{2}}$ and so $N \cong D$, $N \cong E_1 \times Z_p$ or $N \cong Z_{p^2} \times Z_p$ by Theorem 1.3.

Now suppose that $K$ is an extra special $p$-group of order $p^3$. Then $N^{ab} \cong Z_{p}^{(n-1)}$ or $N^{ab} \cong Z_{p^2} \times Z_{p}^{(n-2)}$. If $N^{ab}$ is an elementary abelian, then there is no structure for $N$. Otherwise $N = Z_{p^2}$.

If $m = 4$, then $d(N) = n - 1$. So $N^{ab} \cong Z_{p}^{(n-1)}$ or $N^{ab} \cong Z_{p^2} \times Z_{p}^{n-2}$. In the first case $|N^{ab} \otimes K^{ab}| = |Z_{p}^{(n-1)} \otimes K^{ab}| \leq p^{d(K)(n-1)}$. Now suppose that $d(K) < 4$. Then we have $|N^{ab} \otimes K^{ab}| \leq p^{3(n-1)}$. Therefore $|M(N)| \geq p^{3(n-1) + (8n+n^2-n-10)/2}$ by Corollary 1.2. So $n < 2$ is impossible. If $d(K) = 4$, then we have $K \cong Z_{p}^{(4)}$. Hence $|N^{ab} \otimes K^{ab}| = p^{4(n-1)}$ which implies that $|M(N)| = p^{(n^2-2)/2}$. So $N \cong E_1$ by Theorem 1.3.

In the second case, suppose that $K$ is not abelian. Then $|N^{ab} \otimes K^{ab}| = |Z_{p^2} \times Z_{p}^{(n-2)} \otimes K^{ab}| = |Z_{p^2} \otimes K^{ab}||Z_{p}^{(n-2)} \otimes K^{ab}| \leq p^{3p^{d(K)(n-2)}} \leq p^{3(n-1)}$ which implies that $n < 2$ and it is impossible.

If $K$ is abelian, then $K \cong Z_{p^2} \times Z_{p}^{(2)}$. Thus $|N^{ab} \otimes K^{ab}| = p^{3(n-2)}$. It follows that $N \cong Z_{p^2}$. This completes the proof.
References


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