Generalizations of $\epsilon$-Fixed Point Theorems in Partial Metric Spaces

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Abstract. We consider the dualistic partial metric spaces on a set $X$, and we give necessary conditions for existence of fixed point and $\epsilon$-fixed point for some maps.

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1. Introduction

The partial metric spaces has been introduced by Matthews in [5] as a part of the study of denotational semantics of dataflow networks. In particular, Matthews established the precise relationship between partial metric spaces and the so-called weightable quasi-metric spaces. Indeed he proved a partial metric generalization of Banach contraction mapping theorem.

A partial metric [5] on a set $X$ is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

1. $x = y \iff p(x, x) = p(x, y) = p(y, y)$;
2. $p(x, x) \leq p(x, y)$;

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A partial metric space is a pair \((X,p)\), where \(p\) is a partial metric on \(X\).

If \(p\) is a partial metric on \(X\), then the function \(p^\# : X \times X \to [0,\infty)\) given by \(p^\#(x,y) = 2p(x,y) - p(x,x) - p(y,y)\) is a (usual) metric on \(X\). Each partial metric \(p\) on \(X\) induces a \(T_0\) topology \(\tau_p\) on \(X\) which has as a basis of the family of open \(p\)-balls \(\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}\), where \(B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}\) for all \(x \in X\) and \(\epsilon > 0\). Similarly, closed \(p\)-ball is defined as \(B_p(x,\epsilon) = \{y \in X : p(x,y) \leq p(x,x) + \epsilon\}\).

A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a partial metric space \((X,p)\) is called a Cauchy sequence if there exists (and is finite) \(\lim_{n,m} p(x_n,x_m)\) [5].

A partial metric space \((X,p)\) is said to be complete if every Cauchy sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\) converges, with respect to \(\tau_p\), to a point \(x \in X\) such that \(p(x,x) = \lim_{n,m} p(x_n,x_m)\) [5].

A mapping \(T : X \to X\) is said to be continuous at \(x_0 \in X\), if for \(\epsilon > 0\), there exists \(\delta > 0\) such that \(T(B_p(x_0,\delta)) \subset B_p(T(x_0),\epsilon)\). [1]

**Definition 1.1.** [5] An open ball for a partial metric \(p : X \times X \to [0,\infty)\) is a set of the form \(B_p^\epsilon(x) := \{y \in X : p(x,y) < \epsilon\}\) for each \(\epsilon > 0\) and \(x \in X\).

In [9], S. J. O’Neill proposed one significant change to Matthews definition of the partial metrics, and that was to extend their range from \(R^+\) to \(R\). In the following, partial metrics in the O’Neill sense will be called dualistic partial metrics and a pair \((X,p)\) such that \(X\) is a nonempty set and \(p\) is a dualistic partial metric on \(X\) will be called a dualistic partial metric space.

A dualistic partial metric on a set \(X\) is a function \(p : X \times X \to \mathbb{R}\) such that for all \(x, y, z \in X\):

1. \(x = y \iff p(x,x) = p(x,y) = p(y,y)\);
2. \(p(x,x) \leq p(x,y)\);
3. \(p(x,y) = p(y,x)\);
4. \(p(x,z) \leq p(x,y) + p(y,z) - p(y,y)\).

A dualistic partial metric space is a pair \((X,p)\), where \(p\) is a dualistic partial metric on \(X\).

A quasi-metric on a set \(X\) we mean a nonnegative real-valued function \(d\) on \(X \times X\) such that for all \(x, y, z \in X\):

\[
(3)\quad p(x, y) = p(y, x);
(4)\quad p(x, z) \leq p(x, y) + p(y, z) - p(y, y).
\]
(i) \(d(x, y) = d(y, x) = 0 \iff x = y,\)
(ii) \(d(x, y) \leq d(x, z) + d(z, y).\)

A quasi-metric space is a pair \((X, d)\) such that \(X\) is a (nonempty) set and \(d\) is a quasi-metric on \(X\).

**Lemma 1.2.** [5] If \((X, p)\) is a dualistic partial metric space, then the function \(d_p : X \times X \to \mathbb{R}^+\) defined by \(d_p(x, y) = p(x, y) - p(x, x)\), is a quasi-metric on \(X\) such that \(\tau(p) = \tau(d_p)\).

**Lemma 1.3.** [5] A dualistic partial metric space \((X, p)\) is complete if and only if the metric space \((X, (d_p)^*)\) is complete. Furthermore \(\lim_{n \to \infty} (d_p)^*(a, x_n) = 0\) if and only if \(p(a, a) = \lim_{n \to \infty} p(a, x_n) = \lim_{n, m \to \infty} p(x_n, x_m)\).

Before stating our main results we establish some (essentially known) correspondences between dualistic partial metric spaces and quasi-metric spaces. Our basic references for quasi-metric spaces are [3] and [4] and for \(\epsilon\)-fixed point is [6].

Each quasi-metric \(d\) on \(X\) generates a \(T_0\)-topology \(T(d)\) on \(X\) which has as a base the family of open \(d\)-balls \(B_d(x, \epsilon) := \{y \in X : d(x, y) < \epsilon\}\) for all \(x \in X\) and \(\epsilon > 0\).

If \(d\) is a quasi-metric on \(X\), then the function \(d^*\) defined on \(X \times X\) by \(d^*(x, y) = \max\{d(x, y), d(y, x)\}\), is a metric on \(X\).

**Theorem 1.4.** [6] Let \((X, p)\) be a dualistic partial metric space and \(T : X \to X\) be a map, \(x_0 \in X\) and \(\epsilon > 0\). If \(d_p(T^n(x_0), T^{n+k}(x_0)) \to 0\) as \(n \to \infty\) for some \(k > 0\), then \(T^k\) has an \(\epsilon\)-fixed point.

2. **Main Results**

In this section, we give some results on fixed point and \(\epsilon\)-fixed point in dualistic partial metric space and its diameter.

**Definition 2.1.** An open ball for a dualistic partial metric \(p : X \times X \to \mathbb{R}\) is a set of the form \(B^p(x) := \{y \in X : p(x, y) < \epsilon\}\) for each \(\epsilon > 0\) and \(x \in X\).
Theorem 2.2. Let \((X, p)\) be a dualistic partial metric space and \(K, Y\) be subsets of \(X\). Also, let \(\alpha : K \rightarrow Y\) and \(\beta : Y \rightarrow K\) be two maps. Then 
\(T = \beta \alpha : K \rightarrow K\) has a fixed point if and only if \(S = \alpha \beta : Y \rightarrow Y\), has a fixed point. In other words, given the commutative diagrams:

\[
\begin{align*}
\begin{array}{ccc}
K & \xrightarrow{T} & K \\
\alpha \downarrow & & \downarrow \beta \\
Y & \xrightarrow{\beta} & K
\end{array} & \text{and} & \begin{array}{ccc}
Y & \xrightarrow{\beta} & K \\
\downarrow & & \downarrow \alpha \\
K & \xrightarrow{\alpha} & Y
\end{array}
\end{align*}
\]

we have: \(F(T) \neq \emptyset \iff F(S) \neq \emptyset\).

Proof. If \(y_0\) is a fixed point of \(\beta \alpha\) then it follows that \(\alpha(y_0) = \alpha \beta[\alpha(y_0)]\). \(\Box\)

Definition 2.3. [6] Let \((X, p)\) be a dualistic partial metric space and 
\(T : X \rightarrow X\) be a map. Then \(x_0 \in X\) is \(\epsilon\)-fixed point for \(T\) if 
\[
d_p(Tx_0, x_0) \leq \epsilon.
\]

We say \(T\) has the \(\epsilon\)-fixed point property if for some \(\epsilon > 0\), \(AF(T) \neq \emptyset\) where

\[
AF(T) = \{x_0 \in X : d_p(Tx_0, x_0) \leq \epsilon\}.
\]

Theorem 2.4. Let \((X, p)\) be a dualistic partial metric space and \(K, Y\) be subsets of \(X\). Also, let \(\alpha : K \rightarrow Y\) and \(\beta : Y \rightarrow K\) be two maps and 
\(AF(T) = AF(\alpha)\). Then \(T = \beta \alpha : K \rightarrow K\) has an approximate fixed point if and only if \(S = \alpha \beta : Y \rightarrow Y\), has an approximate fixed point. In other words, given the commutative diagrams:

\[
\begin{align*}
\begin{array}{ccc}
K & \xrightarrow{T} & K \\
\alpha \downarrow & & \downarrow \beta \\
Y & \xrightarrow{\beta} & K
\end{array} & \text{and} & \begin{array}{ccc}
Y & \xrightarrow{\beta} & K \\
\downarrow & & \downarrow \alpha \\
K & \xrightarrow{\alpha} & Y
\end{array}
\end{align*}
\]
we have: \( AF(T) \neq \emptyset \Leftrightarrow AF(S) \neq \emptyset \).

**Proof.** Since \( AF(T) \neq \emptyset \), by Definition 2.3:

\[
d(Ty_0, y_0) \leq \epsilon \Leftrightarrow d(\beta \alpha(y_0), y_0) \leq \epsilon \\
\Leftrightarrow d(\alpha[\beta \alpha(y_0)], \alpha(y_0)) \leq \epsilon \\
\Leftrightarrow d(Sy_0, y_0) \leq \epsilon.
\]

Thus \( AF(T) \neq \emptyset \Leftrightarrow AF(S) \neq \emptyset \). \( \square \)

**Theorem 2.5.** Let \((X, p)\) be a complete dualistic partial metric space and \( T : X \to X \) be a map such that for all \( x, y \in X \)

\[
p(Tx, Ty) \leq cp(x, y) \ ; \ 0 \leq c < 1.
\]

Then \( T \) has a unique fixed point \( u \), and \( T^n(x) \to u \) as \( n \to \infty \) for each \( x \in X \).

**Proof.** We shall show that for any given \( x \in X \), the sequence \( \{T^n(x)\} \) of iterates convergent to a fixed point. For this purpose, first of all observe that \( p(Tx, T^2x) \leq cp(x, Tx) \) and by induction, \( p(T^nx, T^{n+1}x) \leq c^n p(x, Tx) \) for all \( n > 0 \). Thus, for any \( n > 0 \) and any \( k > 0 \), we have

\[
d_p(T^n(x), T^{n+k}(x)) \leq \sum_{i=n}^{n+k-1} d_p(T^i(x), T^{i+1}(x))
\]

\[
\leq (c^n + \cdots + c^{n+k-1}) (p(x, T(x)) - p(x, x))
\]

\[
\leq \frac{c^n}{1-c} (p(x, T(x)) - p(x, x))
\]

\[
= \frac{c^n}{1-c} d_p(x, T(x)).
\]

Since \( c < 1 \), then \( c^n \to 0 \). So by Lemma 1.3, \( \{T^n(x)\} \) is a cauchy sequence in \((X, d_p)\). Hence \( T^n(x) \to u \) for some \( u \in X \). By continuity of \( T \), we should have \( T(T^n(x)) \to Tu \). But \( \{T^{n+1}(x)\} \) is a subsequence of \( \{T^n(x)\} \), so \( Tu = u \) and \( u \) is a fixed point for \( T \). Therefore, we have shown that for each \( x \in X \), the limit of the sequence \( \{T^n(x)\} \) exists.
and is a fixed point, since we will show that $T$ has at most one fixed point, and so every sequence $\{T^n(x)\}$ should be convergent to the same point. At the end we show the uniqueness of the fixed point of $T$: for if $T(x_0) = x_0$ and $T(y_0) = y_0$. Then $x_0 \neq y_0$ gives the contradiction:

$$
\begin{align*}
    d_p(x_0, y_0) &= d_p(T(x_0), T(y_0)) \\
    &= p(T(x_0), T(y_0)) - p(T(x_0), T(x_0)) \\
    &\leq c(p(x_0, y_0) - p(x_0, x_0)) \\
    &< p(x_0, y_0) - p(x_0, x_0) \\
    &= d_p(x_0, y_0).
\end{align*}
$$

\[\square\]

**Corollary 2.6.** Let $(X, p)$ be a complete dualistic partial metric space and $B_p = B_p(y_0, r) = \{y : d_p(y, y_0) < r\}$. Let $T : B_p \to X$ be a map such that for all $x, y \in X$

$$
    p(Tx, Ty) \leq cp(x, y) ; 0 < c < 1.
$$  \(1\)

If $d_p(Ty_0, y_0) < (1 - c)r$, then $T$ has a fixed point.

**Proof.** Choose $\epsilon > r$, so that $d_p(Ty_0, y_0) \leq (1 - c)r < (1 - c)\epsilon$. We show that $T$ maps the closed ball $K = \{y : d_p(y, y_0) \leq \epsilon\}$ into itself: for if $y \in K$, then

$$
\begin{align*}
    d_p(T(y), y_0) &\leq d_p(T(y), T(y_0)) + d_p(T(y_0), y_0) \\
    &\leq cp(y, y_0) + (1 - c)\epsilon \\
    &\leq c\epsilon + \epsilon - c\epsilon = \epsilon.
\end{align*}
$$

Since $K$ is complete and $T : K \to K$ satisfies in (1) thus by Theorem 2.5, $T$ has a fixed point. \[\square\]

**Theorem 2.7.** Let $(X, p)$ be a complete dualistic partial metric space and $T : X \to X$ be a map, not necessarily continuous. For each $\epsilon > 0$ there is a $\theta(\epsilon) > 0$ such that if $d_p(x, Tx) < \theta(\epsilon)$, then $T[B_d(x, \epsilon)] \subset B_d(x, \epsilon)$. Also, if $d_p(T^n(p_0), T^{n+1}(p_0)) \to 0$ for some $p_0 \in X$, the sequence $\{T^n(p_0)\}$ converges to a fixed point of $T$. 
Proof. We consider $T^n(p_0) = x_n$. First we show that $\{x_n\}$ is a Cauchy sequence. since $d_p(x_N, Tx_N) < \theta(\epsilon)$, we have $T[B_d(x_N, \epsilon)] \subset B_d(x_N, \epsilon)$. So $Tx_N = x_{N+1} \in B(x_N, \epsilon)$ and, by induction, $Tx_N = x_{N+k} \in B(x_N, \epsilon)$ for all $k \geq 0$. Thus, $d_p(x_m, x_n) < 2\epsilon$ for all $m, n \geq N$ and $\{x_n\}$ is Cauchy sequence. Therefore it converges to some $x_0 \in X$. Now we show that $x_0$ is a fixed point for $T$. Let $d_p(x_0, Tx_0) = b > 0$, we can choose $x_n \in B(x_0, \frac{b}{3})$ such that $d_p(x_n, x_{n+1}) < \theta\frac{b}{3}$: we have $T[B_d(x_N, \frac{b}{3})] \subset B_d(x_N, \frac{b}{3})$ by hypothesis, so $Tx_0 \in B(x_0, \frac{b}{3})$. But this is impossible because $d_p(Tx_0, x_n) \geq d_p(Tx_0, x_0) - d_p(x_0, x_0) \geq \frac{2b}{3}$. Thus $Tx_0 \notin B(x_0, \frac{b}{3})$ and so $d_p(x_0, Tx_0) = 0$. □

Theorem 2.8. Let $(X, p)$ be a complete dualistic partial metric space and $T : X \to X$ be a map satisfying

$$p(Tx, Ty) \leq \eta|p(x, y)|,$$

where $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is any nondecrasing (not necessarily continuous) function such that $\eta^n(t) \to 0$ as $n \to \infty$ for each $t > 0$. Then $T$ has a unique fixed point $p_0$, and $T^n(x) \to p_0$ as $n \to \infty$ for $x \in X$.

Proof. Observe that $\eta(t) < t$ for each $t > 0$, for if $t \leq \eta(t)$ for some $t > 0$, then monotonicity of $\eta$ gives that $\eta(t) \leq \eta[\eta(t)]$ and by induction, $t \leq \eta^n(t)$ for all $n > 0$. So we have $p(Tx, T^2x) \leq cp(x, Tx)$ and by induction $p(T^n x, T^{n+1} x) \leq c^n p(x, Tx)$ for all $n > 0$. Fix $x \in X$. Then clearly for each $x \in X$

$$|p(T^n x, T^{n+1} x)| \leq c^n|p(x, x)|$$

and

$$|p(T^n x, T^{n+1} x)| \leq c^n|p(x, Tx)|.$$

Also,

$$d_p(T^n x, T^{n+1} x) + p(T^n x, T^n x) = p(T^n x, T^{n+1} x).$$

Hence we deduce that

$$d_p(T^n x, T^{n+1} x) + p(T^n x, T^n x) \leq c^n|p(x, Tx)|.$$
Thus we get
\[ d_p(T^n(x), T^{n+1}(x)) \leq \eta^n |p(x, Tx)| - |p(T^n x, T^n x)| \]
\[ \leq \eta^n |p(x, Tx)| + |p(T^n x, T^n x)| \]
\[ \leq \eta^n (|p(x, Tx)| - |p(x, x)|) \]
\[ \leq \eta^n [d_p(x, Tx)]. \]

So \( d_p(T^n(x), T^{n+1}(x)) \to 0 \) as \( n \to \infty \) for each \( x \in X \). Now let \( \epsilon > 0 \) be given, and choose \( \theta(\epsilon) = \epsilon - \eta(\epsilon) \); if \( d_p(x, Tx) < \theta(\epsilon) \), then for any \( x_0 \in B_d(x, \epsilon) \) we have
\[ d_p(Tx_0, x) \leq d_p(Tx_0, Tx) + d_p(Tx, x) < \eta[d_p(x_0, x)] + \theta(\epsilon) \leq \eta(\epsilon) + \epsilon - \eta(\epsilon) = \epsilon. \]

So \( Tx_0 \in B_d(x, \epsilon) \). Hence by Theorem 2.7, \( T \) has a fixed point. The remainder of the proof is obvious. \( \square \)

**Theorem 2.9.** Let \( (X, p) \) be a complete dualistic partial metric space and \( T : X \to X \) be a map satisfying
\[ p(Tx, Ty) \leq \beta(x, y)p(x, y), \]
where \( \beta : X \times X \to R^+ \) has the property for any closed interval \([a, b] \subset R^+ - \{\nu\}, \)
\[ \sup\{\beta(x, y) : a \leq d_p(x, y) \leq b\} = \lambda(a, b) < 1. \]

Then \( T \) has an unique fixed point \( p \), and \( T^n(x) \to p \) as \( n \to \infty \) for each \( x \in X \).

**Proof.** For \( x \in X \), the sequence \( \{d_p(T^n(x), T^{n+1}(x))\} \) is nonincreasing, therefore it is convergent to some \( a \geq 0 \). We should have \( a = 0 \): otherwise, \( d_p(T^n(x), T^{n+1}(x)) \in [a, a + 1] \) for all large \( n \); then by choosing \( n \) and \( q = \lambda(a, a + 1) \), by induction we have
\[ a \leq d_p(T^{n+k}(x), T^{n+k+1}(x)) \leq q^k d_p(T^n(x), T^{n+1}(x)) \leq q^k (a + 1) \]
for all \( k > 0 \), but \( q < 1 \), and is a contradiction. Now, suppose \( \epsilon > 0 \), \( \lambda = \lambda(\frac{1}{2}, \epsilon) \) and choose \( \theta = \min\{\frac{\epsilon}{2}, \epsilon(1 - \lambda)\} \). Let \( d_p(x, Tx) < \theta(\epsilon) \) and \( x_0 \in B_d(x, \epsilon) \) then
\[ d_p(Tx_0, x) \leq d_p(Tx_0, Tx) + d_p(Tx, x). \]
If $d(x_0, x) < \frac{\varepsilon}{2}$: then
\[
d_p(Tx_0, x) \leq d_p(x_0, x) + d_p(Tx, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;
\]
and if $\frac{\varepsilon}{2} \leq d(x_0, x) < \varepsilon$: then
\[
d_p(Tx_0, x) \leq \beta(x, y)d_p(x_0, x) + d_p(Tx, x) < \lambda \varepsilon + (1 - \lambda) \varepsilon = \varepsilon.
\]
Hence, $T |B_p(x, \varepsilon) \subset B_p(x, \varepsilon)$, and by Theorem 2.7, $T$ has a fixed point. The remainder of the proof is obvious. □

**Proposition 2.10.** Let $(X, p)$ be a dualistic partial metric space and $T : X \to X$ be a map such that $T$ is asymptotic regular, i.e., for all $x \in X$
\[
d_p(T^n(x_0), T^{n+1}(x_0)) \to 0 \text{ as } n \to \infty.
\]
Then $T$ has an $\varepsilon$–fixed point.

**Proof.** Since $d_p(T^n(x_0), T^{n+1}(x_0)) \to 0$ as $n \to \infty$ then for $\varepsilon > 0$, there exists $n_0 > 0$ such that for all $n \geq n_0$,
\[
d_p(T^n(x_0), T^{n+1}(x_0)) < \varepsilon.
\]
Then $d_p(T^{n_0}(x_0), T(T^{n_0}(x_0))) < \varepsilon$. Therefore $T^{n_0}(x_0)$ is an $\varepsilon$–fixed point of $T$. □

**Theorem 2.11.** Let $T$ be a mapping of a dualistic partial metric space $(X, p)$ into itself such that
\[
|p(Tx, Ty)| \leq c|p(x, y)| \quad 0 < c < d(\alpha(y_0), y_0)
\]
for all $x, y \in X$, and $\varepsilon > 0$. Then $T^k$ has an $\varepsilon$–fixed point, for all $k$.

**Proof.** Fix $x \in X$. It is clear that for each $x \in N$
\[
|p(T^n x, T^{n+1}x)| \leq c^n|p(x, x)|
\]
also
\[
|p(T^n x, T^{n+1}x)| \leq c^n|p(x, Tx)|,
\]
and
\[ d_p(T^n x, T^{n+1} x) + p(T^n x, T^n x) = p(T^n x, T^{n+1} x). \]
We deduce that
\[ d_p(T^n x, T^{n+1} x) + p(T^n x, T^n x) \leq c^n |p(x, T x)|. \]
Hence
\[ d_p(T^n x, T^n y) \leq c^n (|p(x, T x)| + |p(x, y)|). \]
Therefore for \( k, n \in \mathbb{N} \)
\[ d_p(T^n x, T^{n+k} x) \leq d_p(T^n x, T^{n+1} x) + \ldots + d_p(T^{n+k} x, T^{n+k} x) \leq (c^n + \ldots + c^{n+k-1})(|p(x, T x)| + |p(x, y)|) \leq \frac{c^n}{1-c} (|p(x, T x)| + |p(x, y)|) \leq \frac{c^n}{1-c} [d_p(x, T x)]. \]
Thus \( \lim_{n \to \infty} d_p(T^{n+k} x, T^n x) = 0 \) as \( n \to \infty \). Therefore by Proposition 2.10 \( T^k \) has an \( \epsilon \)-fixed point. \( \square \)

If we take \( T : X \to X \) in Theorem 2.2 of [7], we have the following corollary.

**Corollary 2.12.** Let \((X, p)\) be a dualistic partial metric space and \( T : X \to X \) be a mapping and \( \epsilon > 0 \). Also, let
\[ d_p(T x, T y) \leq \alpha d_p(x, y) + \beta (d_p(x, T x) + d_p(y, T y)) \]
for all \( x, y \in X \), where \( \alpha, \beta \geq 0 \) and \( \alpha + 2\beta < 1 \). Then \( T \) has an \( \epsilon \)-fixed point.

**Example 2.13.** Let \( X = (-\infty, 2] \), and let \( p \) be the dualistic metric on \( X \) given by
\[ p(x, y) = \max\{x, y\} \]
for all \(x, y \in X\).
Let \(T\) be the mapping from \(X\) into itself defined by \(T(x) = x - 1\), for all \(X = (-\infty, 2]\). It is immediate to see that
\[
p(T(x), T(y)) \leq \frac{1}{2} p(x, y)
\]
for all \(x, y \in X\). However \(T\) does not have any fixed point. But by Proposition 2.10, for some \(\epsilon > 0\), \(T\) has a \(\epsilon\)-fixed point.

**Definition 2.14.** Let \((X, p)\) be a dualistic partial metric space, \(T : X \to X\), be continues map and \(\epsilon > 0\). We define diameter \(AF(T)\) by
\[
diam(AF(T)) = \sup \{d_p(x, y) : x, y \in AF(T)\}.
\]
If we take \(T : X \to X\) in Theorem 2.8 of [7], we have the following corollary.

**Corollary 2.15.** Let \(T : X \to X\) and \(\epsilon > 0\). If there exists \(\alpha \in [0, 1]\) such that for all \(x, y \in X\)
\[
d_p(Tx, Ty) \leq \alpha d_p(x, y),
\]
then
\[
diam(AF(T)) \leq \frac{2\epsilon}{1 - \alpha}.
\]

**References**


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