On Topological Spaces $X$ Determined by the Torsion Elements of $C(X)$

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Abstract. Let $C(X)$ be the ring of real continuous functions on a Tychonoff space $X$ and $T(X)$ be the set of all torsion elements of $C(X)$. We prove that if $X$ and $Y$ are two zero dimensional compact spaces, then $X \simeq Y$ if and only if the rings generated by $T(X)$ and $T(Y)$ are isomorphic.

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1. Introduction

Throughout this paper, all topological spaces $X$ that we consider are Tychonoff and $C(X)$ ($C^*(X)$) stands for the ring of continuous (bounded) real functions on a topological space $X$. Suppose $f \in C(X)$, we denote the set $f^{-1}\{0\}$ by $Z(f)$, its complement by $Coz(f)$, and the collection of all zero-sets in $X$ by $Z(X)$. For undefined terms and notions, see [8]. We denote the group of units of the ring $R$ by $U(R)$. Suppose that $G$ is an abelian group, by $H \leq G$ we mean that $H$ is a subgroup of $G$, by $T(G)$...
we mean the torsion subgroup of $G$. For the sake of simplicity, $U(C(X))$ and $T(U(C(X)))$ will be denoted by $U(X)$ and $T(X)$, respectively. The set \( \{ f \in U(X) : f(x) > 0, \forall x \in X \} \) is denoted by $U^+(X)$. Suppose $f \in U(X)$, we denote the set $f^{-1}\{1\}$ by $e(f)$ and its complement by $Coe(f)$. One can easily see that \{ $e(f) : f \in U(X)$ \} = $Z(X)$. By $Max(G)$ we mean the set of all maximal subgroups of $G$.

In Section 2, we obtain some general facts about $U(X)$. In particular, in the same section we observe that $U(X)$ is direct product of its torsion subgroup $T(X)$ and $U^+(X)$. In Sections 3, we will focus on $T(X)$ and prove, as the main result, that if $X$ and $Y$ are compact zero dimensional spaces, then $X \simeq Y$ if and only if the rings generated by $T(X)$ and $T(Y)$ are isomorphic.

2. Preliminary Results

We first discuss on cardinality of $U(X)$. Let $B_{C^*(0,1)}$ be the unit ball with center 0. Define $\varphi : C(X) \rightarrow B_{C^*(0,1)}$ by $\varphi(f) = \frac{f}{1+|f|}$, then clearly $\varphi$ is one to one. Therefore, for any topological space $X$, we have $|C(X)| = |B_{C^*(0,1)}|$. Now, suppose that $\varphi : B_{C^*(0,1)} \rightarrow U^+(X) \cap C^*(X)$ by $\varphi(f) = f + 2$. It is clear that $\varphi$ is well defined, one-one and thus $|C(X)| = |B_{C^*(0,1)}| \leq |U^+(X) \cap C^*(X)| \leq |U^+(X)| \leq |U(X)| \leq |C(X)|$. Therefore $|U^+(X)| = |U(X)| = |C(X)| = |C^*(X)|$.

Proposition 2.1. The following statements hold.

(a) $T(X) = \{ f \in U(X) : f^2 = 1 \}$ and it is a subgroup of $U(X)$.

(b) The cardinality of the set of torsion free elements is the same as the cardinality of $U(X)$.

(c) $T(X) = \{-1, 1\}$ if and only if $X$ is connected.

Proof. (a) and (b) are clear.

(c $\Rightarrow$) Suppose that $A$ is a clopen subset (i.e., closed open subset) of $X$. Put $\lambda_A = \chi_A - \chi_{A^c}$ (from now on, we use $\lambda_A$ for $\chi_A - \chi_{A^c}$ where $\chi_A$ is the characteristic function on $A$). By hypothesis, $\lambda_A = -1$ or 1. Therefore, $A = \emptyset$ or $A = X$ and consequently $X$ is connected.

(c $\Leftarrow$) Assume that $X$ is connected and $f \in T(X)$. Then $f(X) \subseteq \{-1, 1\}$
and it follows that $f$ is constant. Therefore $f = -1$ or $1$. □

Let $\mathcal{P}$ be the set of all clopen subsets of $X$. Clearly $X$ is zero dimensional if and only if $\mathcal{P}$ is a base for open subset of $X$. Moreover, the map $f \mapsto e(f)$ makes a one-to-one correspondence between $T(X)$ and $\mathcal{P}$ and hence $|T(X)| = |\mathcal{P}|$.

**Proposition 2.2.** Let $\alpha$ be the cardinality of the set of connected component of a topological space $X$. Then
(a) $|T(X)| \leq 2^{\alpha}$ and the inequality may be strict.
(b) If $\alpha$ is finite, then $|T(X)| = 2^\alpha$.

**Proof.** (a) It is enough to show that $|\mathcal{P}| \leq 2^\alpha$. To see this, letting $\mathcal{A}$ be the set of connected component of $X$, we define $\phi: \mathcal{P} \to \mathcal{P}(\mathcal{A})$ with $\phi(P) = \{C \in \mathcal{A} : C \subseteq P\}$. We can easily see that $\phi$ is one-one and so we are done. Now, if we put $X = \mathbb{N}^*$ where $\mathbb{N}^*$ is the one point compactification of $\mathbb{N}$, then the cardinality of the family of clopen subsets of $\mathbb{N}^*$ is equal to $\aleph_0 = |T(X)|$.
(b) It is evident.

The socle $S(G)$ of an abelian group $G$ consists of all $g \in G$ such that the order of $g$ is a square free integer, see [7]. $S(G)$ is a subgroup of $G$; it is equal to $\{1\}$ if and only if $G$ is torsion free and it is equal to $G$ if and only if $G$ is an elementary group, in the sense that every element has a square free order. It is clear that $S(G) \subseteq T(G)$. Therefore, by Proposition 2.1, we conclude that $S(U(X)) = T(X)$. □

The following fact, although easy to prove, is a key result for the remainder of the paper.

**Theorem 2.3.** For any topological space $X$, $U(X)$ is the direct product of $U^+(X)$ and $T(X)$.

**Proof.** It is clear that $f = |f|\text{sgn}(f)$ for any $f \in U(X)$ and $U^+(X) \cap T(X) = \{1\}$. □

**Theorem 2.4.** The following statements hold.
(a) If $K \in \text{Max}(U(X))$, then $U^+(X) \subseteq K$. 

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(b) $K \in \text{Max}(U(X))$ if and only if there exists $H \in \text{Max}(T(X))$ such that $K = U^+(X)H$.

(c) If $K \leq U(X)$, then $K \in \text{Max}(U(X))$ if and only if $|U(X)/K| = 2$.

**Proof.** (a) There exists a prime number $p$ such that $|U(X)/K| = p$, then $f^p \in K$ for any $f \in U(X)$. Now, let $f \in U^+(X)$, then $f = (f^p)^p \in K$.

(b ⇒) Using the part (a) and Theorem 2.3, we get $K = U^+(X)H$ where $H = K \cap T(X)$.

(b ⇐) It is clear.

(c) By assumption, $|U(X)/K|$ is a prime number. On the other hand, by (b), $H \leq T(X)$ exists such that $K = U^+(X)H$, thus $U(X)/K \simeq T(X)/H$ and since every element of $T(X)$ is of order $2$, $|U(X)/K| = 2$. □

Recall that the Frattini subgroup of a group $G$ is the intersection of all maximal subgroups of $G$, this subgroup is denoted by $\Phi(G)$, thus $\Phi(G) = \cap_{H \in \text{Max}(G)} H$.

**Proposition 2.5.** For any topological space $X$ we have

(a) $\Phi(T(X)) = \{1\}$;

(b) $\Phi(U(X)) = U^+(X)$.

**Proof.** (a) Let $1 \neq f \in T(X)$, hence there exists $x \in X$ such that $f(x) \neq 1$. One can easily see that $H_x = \{g \in T(X) : x \in e(g)\} \in \text{Max}(T(X))$ and since $f \notin H_x$, we are through.

(b) It is clear that $\Phi(U(X)) = U^+(X)\Phi(T(X)) = U^+(X)$. □

We conclude this section by the following remark which is useful for the next section and helps us to find an example of two zero dimensional compact spaces $X$ and $Y$ such that $T(X) \simeq T(Y)$ but $X \not\simeq Y$.

**Remark 2.6.** The subgroup $T(X)$ is indeed the maximal torsion subgroup of $U(X)$ and is $\mathbb{Z}_2$-vector space (via $(n,f) \rightarrow f^n$). Clearly $\varphi : T(X) \rightarrow T(Y)$ is a group homomorphism if and only if it is a vector space homomorphism. Let $V$ be a vector space over a field $F$ and $S$ be a base for $V$. If $|F|$ and $|S|$ are finite, then $V \simeq F^{|S|}$ and so $|V| = |F|^{|S|}$. Also, if $|F|$ or $|S|$ is infinite, then $|V| = \max\{|F|,|S|\}$. Therefore, if $V$
and $W$ are $F$-vector spaces and $|V| = |W|$, then $V \simeq W$ whenever one of the following holds.
(a) $F$ is finite.
(b) $|F| < |V|$.

3. Zero Dimensionality is a Torsion Property

To give the main result of the paper we need to introduce and study a class of subgroups of $T(X)$ and $\mathcal{P}$-filters on $X$.

The next two simple facts are needed.

**Proposition 3.1.** The following statements are equivalent.
(a) $H \in \text{Max}(T(X))$.
(b) $fg \in H$ if and only if $f, g \in H$ or $f, g \notin H$.

**Proof.** Since $H \in \text{Max}(T(X))$ if and only if $|T(X)/H| = 2$, it is easy to prove. □

**Lemma 3.2.** Let $f, g \in T(X)$, then
$$e(fg) = (e(f) \cap e(g)) \cup (\text{Coe}(f) \cap \text{Coe}(g)).$$

**Proof.** It is evident. □

**Proposition 3.3.** Let $X$ be a topological space and $p \in \beta X$, then $H^p = \{f \in T(X) : p \in cl_{\beta X}e(f)\}$ is a maximal subgroup of $T(X)$.

**Proof.** Suppose that $fg \in H^p$, by Lemma 3.2
$$p \in cl_{\beta X}e(fg) = cl_{\beta X}[(e(f) \cap e(g)) \cup (\text{Coe}(f) \cap \text{Coe}(g))]$$
$$= (cl_{\beta X}e(f) \cap cl_{\beta X}e(g)) \cup (cl_{\beta X}\text{Coe}(f) \cap cl_{\beta X}\text{Coe}(g)).$$
Thus, $p \in (cl_{\beta X}e(f) \cap cl_{\beta X}e(g))$ or $p \in (cl_{\beta X}\text{Coe}(f) \cap cl_{\beta X}\text{Coe}(g))$ and by Proposition 3.1, $H^p \in \text{Max}(T(X))$. □

In this section, as we mentioned earlier, $\mathcal{P}(X)$ (briefly $\mathcal{P}$) stands for the set of all clopen subsets of $X$ and by $\mathcal{P}$-filter we mean a filter whose
elements are clopen subsets, see ([11] 12E). It is easy to see that if \( F \) is a \( \mathcal{P} \)-filter, then

\[
e^{-1}(\mathcal{F}) = \{ f : e(f) \in \mathcal{F} \}
\]

is a subgroup of \( T(X) \). On the other hand \( ee^{-1}(\mathcal{F}) = \mathcal{F} \) for every \( \mathcal{P} \)-filter \( \mathcal{F} \) on \( X \) and since \( e(f) = e(g) \) implies \( f = g \) for every \( f, g \in T(X) \), \( H = e^{-1}e(H) \) for every \( H \leq T(X) \). But if \( H \) is a subgroup of \( T(X) \), then \( e(H) = \{ e(f) : f \in H \} \) is not necessarily a \( \mathcal{P} \)-filter. As an example, \( H = \{-1, 1\} \) is a subgroup of \( T(X) \) while \( e(H) \) has not even finite intersection property.

**Proposition 3.4.** Let \( X \) be a topological space and \( H \leq T(X) \), then the following statements are equivalent.

(a) There exists \( p \in \beta X \) such that \( H = Hp \).

(b) \( e(H) \) is a \( \mathcal{P} \)-ultrafilter.

(c) The family \( e(H) \) has the finite intersection property and is maximal with respect to this property.

**Proof.** (a)⇒(b) Let \( f_1, \ldots, f_n \in H \). By definition, \( p \in \cap_{i=1}^n \text{cl}_{\beta X} e(f_i) = \text{cl}_{\beta X} (\cap_{i=1}^n e(f_i)) \) and this implies \( \cap_{i=1}^n e(f_i) \neq \emptyset \), thus \( e(H) \) has the finite intersection property, and there exists a \( \mathcal{P} \)-ultrafilter \( \mathcal{F} \) containing \( e(H) \). Therefore, \( H = e^{-1}e(H) \subseteq e^{-1}(\mathcal{F}) \). Now, since \( H \) is maximal, \( H = e^{-1}(\mathcal{F}) \) and hence \( e(H) = ee^{-1}(\mathcal{F}) = \mathcal{F} \).

(b)⇒(c) It is clear.

(c)⇒(a) Suppose that \( H \) satisfies the condition (c). Since \( \beta X \) is compact, there exists \( p \in \beta X \) such that \( p \in \cap_{f \in H} \text{cl}_{\beta X} e(f) \). Clearly \( Hp \) has the finite intersection property and contains \( H \). Therefore, \( H = Hp \). \( \square \)

**Proposition 3.5.** Let \( X \) be a topological space and \( \mathcal{F} \) be a \( \mathcal{P} \)-filter on \( X \), then \( e^{-1}(\mathcal{F}) \) is a maximal subgroup of \( T(X) \) if and only if \( \mathcal{F} \) is a \( \mathcal{P} \)-ultrafilter on \( X \).

**Proof.** \( \Rightarrow \) Let \( e^{-1}(\mathcal{F}) \) be a maximal subgroup of \( T(X) \), we have to show that \( \mathcal{F} \) is a \( \mathcal{P} \)-ultrafilter. Let \( \mathcal{F} \subseteq \mathcal{G} \), then \( e^{-1}(\mathcal{F}) \subseteq e^{-1}(\mathcal{G}) \) and \( e^{-1}(\mathcal{F}) = e^{-1}(\mathcal{G}) \). We infer that \( \mathcal{F} = ee^{-1}(\mathcal{F}) = ee^{-1}(\mathcal{G}) = \mathcal{G} \) and hence \( \mathcal{F} \) is a \( \mathcal{P} \)-ultrafilter.
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\( \Rightarrow \) Since \( ee^{-1}(F) = F \) is a \( \mathcal{P} \)-ultrafilter, by Proposition 3.4, it follows that \( e^{-1}(F) \) is a maximal subgroup of \( T(X) \). \( \square \)

Let \( X \) be a topological space, we will denote by \( M^* \) the set of all maximal subgroups of \( T(X) \) which are of the form \( H^p \). Given \( f \in T(X) \), we define \( M^*(f) = \{ M \in M^* : f \in M \} \) and \( M^*_f = \bigcap M^*(f) \).

**Proposition 3.6.** Let \( X \) be a topological space, then the following statements are equivalent.

(a) \( e(f) \subseteq e(g) \).
(b) \( g \in \bigcap M^*(f) \).
(c) \( M^*(f) \subseteq M^*(g) \).
(d) \( M^*_g \subseteq M^*_f \).

**Proof.** (a \( \Rightarrow \) b) Let \( f \in H^p \in M^*(f) \), then \( p \in cl_{\beta X}e(f) \subseteq cl_{\beta X}e(g) \) and so \( g \in H^p \). Hence \( g \in \bigcap M^*(f) \).
(b \( \Rightarrow \) c) and (c \( \Rightarrow \) d) are trivial.
(d \( \Rightarrow \) a) Let \( x \in e(f) \), then \( f \in H^x \) and so \( g \in M^*_g \subseteq M^*_f \subseteq H^x \). Therefore \( x \in e(g) \). \( \square \)

Recall that a topological space \( X \) is called strongly zero dimensional if every two completely separated closed set can be separated by two disjoint clopen subsets of \( X \), see ([4] 6.2).

In the following proposition, only the part (d) may not be well-known.

**Proposition 3.7.** Let \( X \) be a Tychonoff space. Then the following statements are equivalent.

(a) \( X \) is strongly zero dimensional.
(b) Every two disjoint \( Z_1, Z_2 \in Z(X) \) separate with two disjoint open-closed subset of \( X \).
(c) If \( Z \in Z(X) \), \( V \in Coz(X) \) and \( Z \subseteq V \), then there exists an open-closed subset \( U \) of \( X \) such that \( Z \subseteq U \subseteq V \).
(d) The set \( \{ cl_{\beta X}e(f) : f \in T(X) \} \) is an open base for \( \beta X \).
(e) \( \beta X \) is a zero dimensional space.
(f) \( \beta X \) is a strongly zero dimensional space.

**Proof.** We only prove (c) \( \Rightarrow \) (d), for the reminder of the proof, see ([4] 6.2).
6.2). Suppose that $W$ is an open subset of $\beta X$ and $p \in W$. Clearly there exist a zero set $A$ and a cozero set $V$ in $\beta X$ such that

$$p \in \text{int}_{\beta X} A \subseteq A \subseteq V \subseteq \text{cl}_{\beta X} V \subseteq W.$$  \hfill (1)

By (c), there exists a clopen subset $U$ of $X$ such that

$$A \cap X \subseteq U \subseteq V \cap X.$$ \hfill (2)

Both (1) and (2) implies that

$$p \in \text{cl}_{\beta X} \text{int}_{\beta X} A = \text{cl}_{\beta X} (X \cap \text{int}_{\beta X} A) \subseteq \text{cl}_{\beta X} (X \cap U) \subseteq \text{cl}_{\beta X} V \subseteq W.$$

Clearly $U = e(f)$ for some $f \in T(X)$ and so we are through. \hfill $\Box$

**Proposition 3.8.** Let $X$ be a topological space, then

(a) The set $\{\mathcal{M}^*(f) : f \in T(X)\}$ is a base for the closed sets of a topology on $\mathcal{M}^*$, which we call it the Zariski-like topology;

(b) $\mathcal{M}^*$ with Zariski-like topology is compact;

(c) If $X$ is strongly zero dimensional, then $\mathcal{M}^*$ with this topology is Hausdorff.

**Proof.**

(a) It is clear that for every $f, g \in T(X)$, $e(f) \cup e(g)$ is both open and closed in $X$ and hence there exists $h \in T(X)$ such that $e(h) = e(f) \cup e(g)$. It is enough to show that $\mathcal{M}^*(f) \cup \mathcal{M}^*(g) = \mathcal{M}^*(h)$. To prove this

$$H^p \in \mathcal{M}^*(f) \cup \mathcal{M}^*(g) \iff p \in \text{cl}_{\beta X} e(f) \cup \text{cl}_{\beta X} e(g) = \text{cl}_{\beta X} e(h) \iff H^p \in \mathcal{M}^*(h).$$

(b) Suppose $\{\mathcal{M}^*(f_\alpha)\}_{\alpha \in A}$ is a family of basic closed subset of $\mathcal{M}^*$ with the finite intersection property. We can easily see that $\{\text{cl}_{\beta X} e(f_\alpha)\}_{\alpha \in A}$ has the finite intersection property and consequently there exists $p \in \beta X$ such that $p \in \cap_{\alpha \in A} \text{cl}_{\beta X} e(f_\alpha)$. Therefore

$$\forall \alpha \in A, \ f_\alpha \in H^p \iff \forall \alpha \in A, \ H^p \in \mathcal{M}^*(f_\alpha) \iff H^p \in \bigcap_{\alpha \in A} \mathcal{M}^*(f_\alpha)$$

$$\therefore \bigcap_{\alpha \in A} \mathcal{M}^*(f_\alpha) \neq \emptyset.$$
(c) Let $H^p, H^q \in M^*$ where $p \neq q$. Since $X$ is strongly zero dimensional, by Proposition 3.7, there exists $f \in T(X)$ such that $p \in cl_{\beta X}(f)$, $q \notin cl_{\beta X}(f)$ and so $p \notin cl_{\beta X}(-f)$, $q \notin cl_{\beta X}(-f)$. Therefore, $H^p \notin M^*(-f)$, $H^q \notin M^*(f)$ and $M^*(f) \cup M^*(-f) = M^*$. □

**Proposition 3.9.** Let $X$ be a zero dimensional compact topological space, then $\varphi : X \rightarrow M^*$ defined by $\varphi(p) = H^p$ induces a homeomorphism between $X$ and $M^*$.

**Proof.** It is clear that $\varphi$ is bijection. It is sufficient to show that this function maps a base for $X$ to a base for $M^*$. To this end we write

$$\varphi(e(f)) = \{\varphi(p) \in X : p \in e(f)\}$$

$$= \{H^p \in M^* : p \in e(f)\} = \{H^p \in M^* : f \in H^p\} = M^*(f)$$

which completes the proof. □

**Definition 3.10.** Let $X$ and $Y$ be topological spaces. $T(X)$ and $T(Y)$ are said to be strongly isomorphic if there exists an isomorphism from $T(X)$ to $T(Y)$ such that it maps $M^*(T(X))$ onto $M^*(T(Y))$.

**Proposition 3.11.** If $X$ and $Y$ are zero dimensional compact topological spaces, then $X$ and $Y$ are homeomorphic if only if $T(X)$ and $T(Y)$ are strongly isomorphic.

**Proof.** $\Rightarrow$) Let $\psi : X \rightarrow Y$ be a homeomorphism. We define $\varphi : T(Y) \rightarrow T(X)$ by $\varphi(g) = g \circ \psi$. It can be easily shown that $\varphi$ is an isomorphism. We show that $\varphi$ is onto. Supposing $f \in T(X)$, we put $g = f \circ \psi^{-1}$. It is easy to show that $g \in T(Y)$ and $\varphi(g) = f$. Now, let $p \in X$ and $q = \psi(p)$, then

$$g \in H^q(Y) \iff g(q) = 1 \iff g\psi\psi^{-1}(q) = 1 \iff g\psi(p) = 1$$

$$\iff g \circ \psi \in H^p(X) \iff \varphi(g) \in H^p(X).$$

$\Leftarrow$) By Proposition 3.9 and the hypothesis, it is clear that

$$X \simeq M^*(T(X)) \simeq M^*(T(Y)) \simeq Y.$$  □
Proposition 3.12. If \( X \) and \( Y \) are topological spaces, then the following statements are equivalent.

(a) \( T(X) \cong T(Y) \).
(b) \( |T(X)| = |T(Y)| \).
(c) \( |\mathcal{P}(X)| = |\mathcal{P}(Y)| \).

Proof. (a) \( \iff \) (b) is clear, by Remark 2.6. Recall that \( |T(X)| = |\mathcal{P}(X)| \) for any topological space \( X \), so (b) \( \iff \) (c) is also clear. \( \square \)

Example 3.13. Let \( \mathbb{Q}^* \) be the one point compactification of the \( \mathbb{Q} \), \( \omega_1 \) be the smallest uncountable ordinal and \( W^* = \{ \alpha : \alpha \text{ is an ordinal and } \alpha \leq \omega_1 \} \). One can easily see that \( |T(\mathbb{Q}^*)| = \mathfrak{c} = |T(W^*)| \) and by Proposition 3.12, \( T(\mathbb{Q}^*) \cong T(W^*) \) while \( |\mathbb{Q}^*| \neq |W^*| \). Now, suppose \( |A| = \mathfrak{c} \), \( X = \bigcup_{\alpha \in A} X_\alpha \) is the disjoint union of copies of \( \mathbb{Q} \), and \( X^* \) is the one point compactification of \( X \). Then clearly \( |X^*| = |W^*| \) and one can similarly show that \( T(X^*) \cong T(W^*) \) while \( X^* \not\cong W^* \). These examples show that \( T(X) \) and \( T(Y) \) can be isomorphic but not strongly isomorphic (even if \( |X| = |Y| \)). Note that if \( T(X) \) and \( T(Y) \) are strongly isomorphic, then it may there exists an isomorphism between them which is not strong; it is sufficient to define an isomorphism such that sends \( f \neq -1 \) to \(-1\).

Definition 3.14. We say that a subgroup \( H \) of \( T(X) \) is saturated if \( f \in H \) and \( e(f) \subseteq e(g) \) imply that \( g \in H \).

Lemma 3.15. Let \( H \leq T(X) \) be saturated. Then \( e(H) \) is closed under finite intersection.

Proof. Let \( f, g \in H \); it is enough to show that \( e(f) \cap e(g) \in e(H) \). For simplicity we let \( A = e(f) \) and \( B = e(g) \), then by assumption

\[
D = e(fg) \cup (A \setminus B) = (B \setminus A)^c \in e(H).
\]

Therefore, \( \lambda_D \in H \) and hence \( g\lambda_D \in H \). Thus, \( A \cap B = e(g\lambda_D) \in e(H) \). \( \square \)

Corollary 3.16. Let \( H \leq T(X) \), then \( e(H) \) is a \( \mathcal{P} \)-filter if and only if \( H \) is saturated and \(-1 \notin H \).
Corollary 3.17. A subgroup $H$ of $T(X)$ is saturated if and only if $h \in H$ and $1 + h + f \in T(X)$ imply $-f \in H$.

Proof. $\Rightarrow$ Suppose $h \in H$ and $1 + h + f \in T(X)$. It is clear that $e(h) \cap e(f) = \emptyset$ and hence $e(h) \subseteq e(-f)$. Thus $-f \in H$.

$\Leftarrow$ Let $h \in H$ and $e(h) \subseteq e(f)$, thus $e(h) \cap e(-f) = \emptyset$. It is easy to see that $1 + h - f \in T(X)$ and hence $f \in H$. □

Corollary 3.18. $e(H)$ is a $\mathcal{P}$-filter if and only if $-1 \notin H$ and if $h \in H$ and $1 + h + f \in T(X)$, then $-f \in H$.

Proof. By Corollaries 3.16 and 3.17, it is clear. □

Definition 3.19. Let $X$ be a topological space. We will denote by $RT(X)$ the ring generated by $T(X)$.

Proposition 3.20. If

$$ R = \{ f \in C(X), f(X) \text{ is finite, and } f(X) \subseteq 2\mathbb{Z} \text{ or } f(X) \subseteq 2\mathbb{Z} + 1 \}, $$

then $RT(X) = R$.

Proof. One can easily see that $R$ is indeed a ring that contains $T(X)$ and consequently $RT(X) \subseteq R$. Now, let $f \in R$ i.e., $f \in C(X)$ and $f(X) = \{ m_1, \ldots, m_k \}$ is a subset of $2\mathbb{Z}$ or $2\mathbb{Z} + 1$. We have to show $f \in RT(X)$. It is clear that $A_i = f^{-1}\{ m_i \}$ is a clopen subset of $X$. Thus it is enough to prove that the equation $f = x_1 + \sum_{i=2}^{k} x_i A_i$ has a solution for $x_1, \ldots, x_k$ in $\mathbb{Z}$. If we take $a_i \in A_i$ ($i = 1, \ldots, k$), then we get the following equations

$$ \begin{cases} 
    x_1 - x_2 - x_3 - \cdots - x_k = m_1 \\
    x_1 + x_2 - x_3 - \cdots - x_k = m_2 \\
    \vdots \\
    x_1 - x_2 - x_3 - \cdots + x_k = m_k 
\end{cases} $$

and we simply obtain the equivalent system of equations below.

$$ \begin{cases} 
    x_1 - x_2 - x_3 - \cdots - x_k = m_1 \\
    x_2 = \frac{m_2 - m_1}{2} \\
    \vdots \\
    x_k = \frac{m_k - m_1}{2} 
\end{cases} $$
Now, since all the elements of the set \( \{m_1, \ldots, m_k\} \) are even or all are odd, the above system has a solution in \( \mathbb{Z} \). \( \square \)

In [5] and [6] it is defined that 
\[
C_C(X) = \{ f \in C(X) : f(X) \text{ is countable} \}
\]
and 
\[
C^F(X) = \{ f \in C(X) : |f(X)| < \infty \}.
\]
Clearly the ring \( R \) in Proposition 3.20, is a subring of \( C^F(X) \). The following result is proved in [6].

**Theorem 3.21.** Let \( X \) be a topological space. Then there exists a zero dimensional space \( Y \) such that 
\[
C_C(X) \simeq C_C(Y) \quad (C^F(X) \simeq C^F(Y)).
\]

The above theorem shows that, without loss of generality one may assume that \( X \) is a zero dimensional space. Therefore, considering this comment, the following fact which is our main result is in order.

**Theorem 3.22.** Let \( X \) and \( Y \) be compact zero dimensional spaces. Then 
\[
RT(X) \simeq RT(Y) \quad \text{if and only if} \quad X \simeq Y.
\]

**Proof.** \( \Rightarrow \) Suppose \( \varphi : RT(X) \rightarrow RT(Y) \) is an isomorphism. It is sufficient to show that \( \varphi(T(X)) = T(Y) \) and \( \varphi \) maps the set \( M^*(X) \) onto \( M^*(Y) \). It is clear that \( \varphi(T(X)) \subseteq T(Y) \). Now, let \( g \in T(Y) \), then \( f \in RT(X) \) exists such that \( \varphi(f) = g \). Therefore, since \( \varphi \) is one-one, we can write \( \varphi(f^2) = (\varphi(f))^2 = g^2 = 1 \) which implies \( f^2 = 1 \) and therefore \( f \in T(X) \). Now, we have to prove that \( \varphi(M^*(X)) = M^*(Y) \). To this end it is sufficient to show that \( e(H^p(X)) \) is a \( P \)-filter if and only if \( e(\varphi(H^p(X))) \) is such. It is clear that
\[
-1 \notin H^p \iff -1 = \varphi(-1) \notin \varphi(H^p).
\]

On the other hand
\[
f \in H^p, \quad 1 + f + g \in T(X) \iff \varphi(f) \in \varphi(H^p), \quad \varphi(1 + f + g) = 1 + \varphi(f) + \varphi(g) \in T(Y).
\]
It is also clear that \( -g \in H^p \) if and only if \( -\varphi(g) \in \varphi(H^p) \). Hence, by Corollary 3.18, we are through.

\( \Leftarrow \) It is obvious. \( \square \)
Corollary 3.23. $T(X)$ and $T(Y)$ are strongly isomorphic if and only if $RT(X) \simeq RT(Y)$.

References


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