On the normality of directed Cayley graphs of abelian groups with valency four
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Abstract

A Cayley digraph $\Gamma = \text{Cay}(G, S)$ is called normal for $G$ if the right regular representation $R(G)$ of $G$ is normal in the full automorphism group $\text{Aut}(\Gamma)$. In this paper, we determine some non-normal directed Cayley graphs of finite abelian groups with valency 4.

Keywords: Cayley graph, Automorphism groups, Normal Cayley digraph.

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1 Introduction

Throughout this paper graphs are finite, simple and directed. For a graph $\Gamma$, let $V(\Gamma)$, $E(\Gamma)$ and $\text{Aut}(\Gamma)$ denote its vertex set, edge set and full automorphism group, respectively. Let $G$ be a finite group and $S$ a subset of $G$ not containing the identity $1_G$. The Cayley digraph $\Gamma = \text{Cay}(G, S)$ of $G$ with respect to $S$ is a digraph defined by $V(\Gamma) = G$, $E(\Gamma) = \{(g, sg) | g \in G, s \in S\}$. In particular, if $S = S^{-1}$ such a digraph can be viewed as an undirected graph by coalescing each pair $(g, sg)$ and $(sg, g)$ of directed edges into a single undirected edge $\{g, sg\}$. A Cayley digraph $\Gamma = \text{Cay}(G, S)$ is called normal for $G$ if the right regular representation $R(G)$ of $G$ is normal of the

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automorphism group of \( \Gamma \) (see [6]). Also \( \Gamma \) is said to be normal edge-transitive if \( N_{\text{Aut}(X)}(G) \) is transitive on edges.

The concept of normality of Cayley digraphs is known to be of fundamental importance for the study of arc-transitive graphs. So, for a given finite group \( G \), a natural problem is to determine all the normal or non-normal Cayley digraphs of \( G \). Recently, some meaningful results in this direction, especially for the undirected Cayley graphs, have been obtained. For example, [3, 4] determined all non-normal Cayley graphs of group of order \( 2p \) and \( pq \) (\( p \) and \( q \) are prime), respectively. [1] determined all non-normal Cayley graphs of abelian groups with valency at most 4. Later, [2] deal with the case of valency 5. Moreover, some of these results have been applied to the classifications of graphs of certain category (Find more information in [7, 8] for example).

Let \( G \) be a finite abelian group, and \( \Gamma = \text{Cay}(G,S) \) be a connected directed Cayley graph of \( G \) with respect to \( S \) of valency 4. So we may assume that \( S = \{a, b, c, d\} \). We shall divide the set \( S \) into five different cases: (I) \( a^2 = b^2 = c^2 = d^2 \); otherwise, without loss of generality, we may assume that (II) \( a^2 = b^2 \neq c^2 = d^2 \); (III) \( a^2 = b^2 = c^2 \neq d^2 \); (IV) \( a^2 = b^2 = c^2 = d^2 \); (V) \( a^2, b^2, c^2 \) and \( d^2 \) are mutually unequal.

In this paper, we consider the cases (I) and (II) and the other cases are open problems. The following theorem deal with the cases (I) and (II).

**Theorem 1.1** Let \( \Gamma = \text{Cay}(G,S) \) be a connected directed Cayley graph of an abelian group \( G \) with respect to \( S \) of valency 4, and let \( S = \{a, b, c, d\} \). Then:

(i) If \( a^2 = b^2 = c^2 = d^2 \), then \( \Gamma \) is normal.

(ii) If \( a^2 = b^2 \neq c^2 = d^2 \) and \( \langle a, b \rangle \cap \langle c, d \rangle = 1 \), then \( \Gamma \) is normal except one of the following cases happens:

1. \( G = \langle a \rangle \times \langle c \rangle = Z_{2n} \times Z_{2m}, S = \{a, a^{n+1}, c, c^{m+1}\} \) (\( n > 2, m > 2 \)).
2. \( G = \langle a \rangle \times \langle u \rangle \times \langle c \rangle \times \langle v \rangle = Z_n \times Z_2 \times Z_m \times Z_2, S = \{a, au, c, cv\} \), where \( u = ab^{-1} \) and \( v = cd^{-1} \).
3. \( G = \langle a \rangle \times \langle c \rangle \times \langle v \rangle = Z_{2n} \times Z_m \times Z_2, S = \{a, a^{n+1}, c, cv\} \) (\( n > 2 \)), where \( v = cd^{-1} \).
4. \( G = \langle a \rangle \times \langle u \rangle \times \langle c \rangle = Z_n \times Z_2 \times Z_{2m}, S = \{a, au, c, c^{m+1}\} \) (\( m > 2 \)).
2 Preliminaries

In this section, we give some preliminary results to be used in the sequel. Let \( \Gamma = \text{Cay}(G, S) \) be a directed Cayley graph of \( G \) with respect to \( S \) and \( \text{Aut}(G, S) = \{ \alpha \in \text{Aut}(G)|S^\alpha = S \} \). Write \( A := \text{Aut}(\Gamma) \) and denote by \( A_1 \) the subgroup of \( \text{Aut}(\Gamma) \) consisting of all automorphisms which fix the identity \( 1_G \) of \( G \). The following proposition is basic.

**Proposition 2.1** ([2, Proposition 2.1]) Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley (di)graph of \( G \) on the set \( S \).

(i) \( \text{Aut}(\Gamma) \) contains the right regular representation \( R(G) \) of \( G \) and so \( \Gamma \) is vertex-transitive.

(ii) \( \Gamma \) is connected if and only if \( G = \langle S \rangle \).

(iii) \( \Gamma \) is undirected if and only if \( S^{-1} = S \).

Hence, all Cayley (di)graphs are vertex-transitive.

**Proposition 2.2** ([6, Proposition 1.3]) As the above notations:

(i) \( N_A(R(G)) = R(G)\text{Aut}(G, S) \).

(ii) \( A = R(G)\text{Aut}(G, S) \) if and only if \( R(G) \) is normal in \( A \).

**Proposition 2.3** ([6, Proposition 1.5]) The Cayley (di)graph is normal if and only if \( A_1 = \text{Aut}(G, S) \). In particular, [2] gives a sufficient condition for the normality of Cayley (di)graphs of abelian groups.

**Theorem 2.4** ([2, Theorem 1.1]) Let \( G \) be a finite abelian group and \( S \) a generating set of \( G \) not containing the identity \( 1_G \). Assume \( S \) satisfies the condition

\[
  s, t, u, v \in S \text{ with } st = uv \neq 1 \Rightarrow \{s, t\} = \{u, v\}.
\]

Then the Cayley (di)graph \( \text{Cay}(G, S) \) is normal.

We also need the following two simple proposition.
Proposition 2.5 ([2, Proposition 1.5]) Let $G$ be a finite group, $S$ a generating set of $G$ not containing the identity $1_G$, and $\alpha$ an automorphism of $G$. The Cayley (di)graph $\text{Cay}(G, S)$ is normal if and only if $\text{Cay}(G, S^\alpha)$ is normal.

Proposition 2.6 ([2, Proposition 2.6]) Let $G = G_1 \times G_2$ be the direct product of two finite group $G_1$ and $G_2$, let $S_1$ and $S_2$ be subset of $G_1$ and $G_2$ respectively, and let $S = S_1 \cup S_2$ be the disjoint union of two subset $S_1$ and $S_2$. Then we have:

(i) $\text{Cay}(G, S) \cong \text{Cay}(G_1, S_1) \times \text{Cay}(G_2, S_2)$.

(ii) If $\text{Cay}(G, S)$ is normal, then $\text{Cay}(G_1, S_1)$ is normal.

(iii) If $\text{Cay}(G_1, S_1)$ and $\text{Cay}(G_2, S_2)$ are both normal and relatively prime, then $\text{Cay}(G, S)$ is normal.

3 Main Results

Let $G$ be a finite abelian group, and let $\Gamma = \text{Cay}(G, S)$ be a connected directed Cayley graph of $G$ with respect to $S$ of valency 4.

Proof of Theorem 1.1 (i). First of all, we prove that $A_1$ acts faithfully on the set $S$. By the vertex-transitivity and connectedness of $\Gamma$, it suffices to prove that, for any vertex $x$ of $\Gamma$, if an automorphism $\alpha$ of $\Gamma$, fixes $x$ and all elements of $xS$, then $\alpha$ also fixes all elements of $xS^2$. Actually, if $\alpha$ fixes $xS$ pointwise, then for any $v \in xS$, $\alpha$ fixes $vS$ setwise. Hence $(xaS \cap xbS \cap xcS \cap xdS)^\alpha = (xaS)^\alpha \cap (xbS)^\alpha \cap (xcS)^\alpha \cap (xdS)^\alpha = xaS \cap xbS \cap xcS \cap xdS$. That is, $(xa^2)^\alpha = xa^2$. Since $(xaS \cap xbS)^\alpha = xaS \cap xbS$, we have $\{xa^2, xab\}^\alpha = \{xa^2, xab\}$, and as $(xa^2)^\alpha = xa^2$, we have $(xab)^\alpha = xab$. Note that $(xaS \cap xcS)^\alpha = xaS \cap xcS$ and $(xa^2)^\alpha = xa^2$, hence we have $(xac)^\alpha = xac$. The same argument as above can be employed to show that $(xad)^\alpha = xad$, $(xbc)^\alpha = xbc$, $(xbd)^\alpha = xbd$ and $(xcd)^\alpha = xcd$. It now follows that $\alpha$ fixes all elements of $xS^2$.

Now we turn to the normality of $\Gamma$. By Proposition 2.3, what remains to be proved is $A_1 \leq \text{Aut}(G)$. Since $A_1$ acts faithfully on $S$, $A_1$ is isomorphic to a subgroup of $S_4$. For any $\sigma \in A_1$, we prove that $\sigma$ is also in $\text{Aut}(G)$ as follows.
Case 1. $o(\sigma) = 1$. In this case, $\sigma$ is obviously in $\text{Aut}(G)$.

Case 2. $o(\sigma) = 2$. Without loss of generality, we may assume that $a^\sigma = b, b^\sigma = a, c^\sigma = c$ and $d^\sigma = d$. To prove $\sigma \in \text{Aut}(G)$, it suffices to show that

$$(s_1 s_2 \ldots s_n)^\sigma = s_1^\sigma s_2^\sigma \ldots s_n^\sigma, \forall s_1, s_2, \ldots, s_n \in S.$$  \hspace{1cm} (2)

Let $x$ be given in $G$, to prove (2), we only need to show

$$(xs)^\sigma = x^\sigma s^\sigma, \forall s \in S \Rightarrow (xuv)^\sigma = x^\sigma u^\sigma v^\sigma, \forall u, v \in S.$$  \hspace{1cm} (3)

In fact, once (3) is proved, (2) is easily verified by induction on $n$. Suppose $(xs)^\sigma = x^\sigma s^\sigma$ for all $s \in S$. Note that $(gS)^\sigma = g^\sigma S, \forall g \in G$. Since,

$$(xaS \cap xbS \cap xcS \cap xdS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma \cap (xcS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S \cap x^\sigma c^\sigma S \cap x^\sigma d^\sigma S = x^\sigma(bS \cap x^\sigma aS \cap x^\sigma cS \cap x^\sigma dS),$$

we have $(xa^2)^\sigma = x^\sigma b^2 = x^\sigma(a^\sigma)^2$.

Hence $(xb^2)^\sigma = x^\sigma(b^\sigma)^2$, $(xc^2)^\sigma = x^\sigma(c^\sigma)^2$, $(xd^2)^\sigma = x^\sigma(d^\sigma)^2$.

Since

$$(xaS \cap xbS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S = x^\sigma bS \cap x^\sigma aS, \quad \{xa^2, xab\} = \{x^\sigma a^2, x^\sigma ab\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma b^\sigma\}.$$  \hspace{1cm} (4)

However, we have known that $(xa^2)^\sigma = x^\sigma(a^\sigma)^2$. Thus $(xab)^\sigma = x^\sigma a^\sigma b^\sigma$. As $(xaS \cap xcS)^\sigma = (xaS)^\sigma \cap (xcS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma c^\sigma S = x^\sigma bS \cap x^\sigma cS$ and $(xa^2)^\sigma = x^\sigma(a^\sigma)^2$, we have $(xac)^\sigma = x^\sigma a^\sigma c^\sigma$.

Since $(xaS \cap xdS)^\sigma = (xaS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma d^\sigma S = x^\sigma bS \cap x^\sigma dS$, it follows that

$$(x^2a, xad)^\sigma = \{x^\sigma a^2, x^\sigma ad\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma d^\sigma\}, \quad \text{and} \quad (xa^2)^\sigma = x^\sigma(a^\sigma)^2.$$  \hspace{1cm} (5)

We have $(xad)^\sigma = x^\sigma a^\sigma d^\sigma$. Since $(xbS \cap xcS)^\sigma = x^\sigma b^\sigma S \cap x^\sigma c^\sigma S = x^\sigma aS \cap x^\sigma cS$, we conclude that

$$(xb^2, xbc)^\sigma = \{x^\sigma a^2, x^\sigma ac\} = \{x^\sigma(b^\sigma)^2, x^\sigma b^\sigma c^\sigma\}.$$  \hspace{1cm} (6)

But we have already had $(xb^2)^\sigma = x^\sigma(b^\sigma)^2$, Thus $(xbc)^\sigma = x^\sigma b^\sigma c^\sigma$. The same argument as above can be employed to show that $(xbd)^\sigma = x^\sigma b^\sigma d^\sigma$ and $(xcd)^\sigma = x^\sigma c^\sigma d^\sigma$.

Case 3. $o(\sigma) = 2$. But in this case, we may assume that $a^\sigma = b, b^\sigma = a, c^\sigma = d$, and $d^\sigma = c$. As above, we only need to prove (3). Suppose $(xs)^\sigma = x^\sigma s^\sigma$ for all $s \in S$.

Since

$$(xaS \cap xbS \cap xcS \cap xdS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma \cap (xcS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S \cap x^\sigma c^\sigma S \cap x^\sigma d^\sigma S = x^\sigma(bS \cap x^\sigma aS \cap x^\sigma dS \cap x^\sigma cS),$$

we have $(xa^2)^\sigma = x^\sigma b^2 = x^\sigma(a^\sigma)^2$.

Hence $(xb^2)^\sigma = x^\sigma(b^\sigma)^2$, $(xc^2)^\sigma = x^\sigma(c^\sigma)^2$, $(xd^2)^\sigma = x^\sigma(d^\sigma)^2$.
\[(xaS \cap xbS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S = x^\sigma bS \cap x^\sigma aS, \{xa^2, xab\}^\sigma = \{x^\sigma a^2, x^\sigma ab\} = \{x^2b^2, x^\sigma a^\sigma b^\sigma\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma b^\sigma\}.\] However, we have known that \((xa^2)^\sigma = x^\sigma(a^\sigma)^2\). Thus \((xab)^\sigma = x^\sigma a^\sigma b^\sigma\). As \((xaS \cap xcS)^\sigma = (xaS)^\sigma \cap (xcS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma c^\sigma S = x^\sigma bS \cap x^\sigma dS\) and \((xa^2)^\sigma = x^\sigma(a^\sigma)^2\), we have \((xac)^\sigma = x^\sigma a^\sigma c^\sigma\). Since \((xaS \cap xdS)^\sigma = (xaS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma d^\sigma S = x^\sigma bS \cap x^\sigma cS\), it follows that \(\{xa^2, xdS\}^\sigma = \{x^\sigma a^2, x^\sigma a^\sigma d^\sigma\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma d^\sigma\}\), and as \((xa^2)^\sigma = x^\sigma(a^\sigma)^2\), we have \(\{xa^2, xdS\}^\sigma = x^\sigma a^\sigma d^\sigma\). Since \((xbS \cap xcS)^\sigma = x^\sigma b^\sigma S \cap x^\sigma c^\sigma S = x^\sigma aS \cap x^\sigma dS\), we conclude that \(\{xb^2, xbc\}^\sigma = \{x^\sigma a^2, x^\sigma ac\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma c^\sigma\}\). But we have already had \((xb^2)^\sigma = x^\sigma(b^\sigma)^2\), thus \((xbc)^\sigma = x^\sigma b^\sigma c^\sigma\). The same argument as above can be employed to show that \((xbd)^\sigma = x^\sigma b^\sigma d^\sigma\) and \((xcd)^\sigma = x^\sigma c^\sigma d^\sigma\).

**Case 4.** \(o(\sigma) = 3\). In this case, we assume that \(a^\sigma = b, b^\sigma = c, c^\sigma = a\) and \(d^\sigma = d\). As above, we only need to prove (3). Suppose \((xs)^\sigma = x^\sigma s^\sigma\) for all \(s \in S\). Since,

\[(xaS \cap xbS \cap xcS \cap xdS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma \cap (xcS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S \cap x^\sigma c^\sigma S \cap x^\sigma d^\sigma S = x^\sigma bS \cap x^\sigma cS \cap x^\sigma dS, \{xa^2, xab\}^\sigma = \{x^\sigma a^2, x^\sigma ab\} = \{x^\sigma(b^\sigma)^2, x^\sigma a^\sigma b^\sigma\}\). However, we have known that \((xa^2)^\sigma = x^\sigma(a^\sigma)^2\). Thus \((xab)^\sigma = x^\sigma a^\sigma b^\sigma\). As \((xaS \cap xcS)^\sigma = (xaS)^\sigma \cap (xcS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma c^\sigma S = x^\sigma bS \cap x^\sigma aS\) and \((xa^2)^\sigma = x^\sigma(a^\sigma)^2\), we have \((xcac)^\sigma = x^\sigma a^\sigma c^\sigma\). Since \((xaS \cap xdS)^\sigma = (xaS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma d^\sigma S = x^\sigma bS \cap x^\sigma dS\), it follows that \(\{xa^2, xdD\}^\sigma = \{x^\sigma b^2, x^\sigma bd\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma d^\sigma\}\), and as \((xa^2)^\sigma = x^\sigma(a^\sigma)^2\), we have \((xdad)^\sigma = x^\sigma a^\sigma d^\sigma\). Since \((xbS \cap xcS)^\sigma = x^\sigma b^\sigma S \cap x^\sigma c^\sigma S = x^\sigma cS \cap x^\sigma aS\), we conclude that \(\{xb^2, xbc\}^\sigma = \{x^\sigma a^2, x^\sigma ac\} = \{x^\sigma(b^\sigma)^2, x^\sigma b^\sigma c^\sigma\}\). But we have already had \((xb^2)^\sigma = x^\sigma(b^\sigma)^2\), thus \((xbc)^\sigma = x^\sigma b^\sigma c^\sigma\). The same argument as above can be employed to show that \((xbd)^\sigma = x^\sigma b^\sigma d^\sigma\) and \((xcd)^\sigma = x^\sigma c^\sigma d^\sigma\).

**Case 5.** \(o(\sigma) = 4\). In this case, we may assume that \(a^\sigma = b, b^\sigma = c, c^\sigma = d\) and \(d^\sigma = a\). As above, we only need to prove (3). Suppose \((xs)^\sigma = x^\sigma s^\sigma\) for all \(s \in S\). Since,

\[(xaS \cap xbS \cap xcS \cap xdS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma \cap (xcS)^\sigma \cap (xdS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S\]
Hence \((a^2)^2 = x^2b^2 = x^2(a^2)^2\).

Since
\[
\{xa^2, x^2bS \cap x^2cS \cap x^2dS \cap x^2aS, \text{ we have } (a^2)^2 = x^2b^2 = x^2(a^2)^2.\]
\[
\text{Hence } (xb^2)^2 = x^2(b^2)^2, (xc^2)^2 = x^2(c^2)^2, (xd^2)^2 = x^2(d^2)^2.\]

Since,
\[
(xaS \cap xbS)^\sigma = (xaS)^\sigma \cap (xbS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma b^\sigma S = x^\sigma bS \cap x^\sigma cS, \{xa^2, xab\} = \{x^\sigma a^2, x^\sigma bc\} = \{x^\sigma b^2, x^\sigma a^\sigma b^\sigma\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma b^\sigma\}. \]

However, we have known that
\[
(xa^2)^\sigma = x^\sigma(a^\sigma)^2. \]

Thus \((xab)^\sigma = x^\sigma a^\sigma b^\sigma\). As \((xaS \cap xcS)^\sigma = (xaS)^\sigma \cap (xcS)^\sigma = x^\sigma a^\sigma S \cap x^\sigma cS \cap x^\sigma S = x^\sigma bS \cap x^\sigma aS, \text{ it follows that } \{xa^2, xad\}^\sigma = \{x^\sigma a^2, x^\sigma ba\} = \{x^\sigma(a^\sigma)^2, x^\sigma a^\sigma d^\sigma\}, \text{ and as } (a^2)^\sigma = x^\sigma(a^\sigma)^2, \text{ we have } (xad)^\sigma = x^\sigma a^\sigma d^\sigma. \]

Since \((xbS \cap xcS)^\sigma = x^\sigma b^\sigma S \cap x^\sigma c^\sigma S \subseteq x^\sigma cS \cap x^\sigma dS, \text{ we conclude that } \{xb^2, xbc\}^\sigma = \{x^\sigma a^2, x^\sigma cd\} = \{x^\sigma(b^\sigma)^2, x^\sigma b^\sigma c^\sigma\}. \]

But we have already had \((xb^2)^\sigma = x^\sigma(b^\sigma)^2\), thus \((xbc)^\sigma = x^\sigma b^\sigma c^\sigma. The same argument as above can be employed to show that \((xbd)^\sigma = x^\sigma b^\sigma d^\sigma \text{ and } (xcd)^\sigma = x^\sigma c^\sigma d^\sigma. \)

So we come to the conclusion that \(A_1 \leq Aut(G). \]

**Proof of Theorem 1.1 (ii).** Since \((a, b) \cap (c, d) = 1 \text{ and } G = \langle S \rangle = \langle a, b, c, d \rangle, \text{ we have } G = \langle a, b \rangle \times \langle c, d \rangle. \]

By Proposition 2.6(i), it is easy to see that \(\Gamma \cong Cay(\langle a, b \rangle, \{a, b\}) \times Cay(\langle c, d \rangle, \{c, d\}) = \Gamma_1 \times \Gamma_2. \]

Since \(a^2 = b^2, \text{ by [9, Lemma 3.1], we know that } \Gamma_1 \text{ is non-normal, and we can also see that } \Gamma \text{ is non-normal by Proposition 2.6(ii). In this case, } G = \langle a, u \rangle \times \langle c, v \rangle \text{ where } u = ab^{-1} \text{ and } v = cd^{-1} \text{ are both involution.} \)

If \(u \notin \langle a \rangle \text{ and } v \notin \langle c \rangle, \text{ then } G = \langle a \rangle \times \langle c \rangle = Z_{2n} \times Z_{2m}(n > 2, m > 2 \text{ for some non-negative integer } n, m) \text{ and } S = \{a, a^{n+1}, c, c^{m+1}\}. \]

If \(u \notin \langle a \rangle \text{ and } v \notin \langle c \rangle, \text{ then } G = \langle a \rangle \times \langle u \rangle \times \langle x \rangle \times \langle v \rangle = Z_n \times Z_2 \times Z_m \times Z_2, S = \{a, au, c, cv\}. \]

If \(u \notin \langle a \rangle \text{ and } v \notin \langle c \rangle, \text{ then } G = \langle a \rangle \times \langle c \rangle \times \langle x \rangle = Z_{2n} \times Z_2 \times Z_2(n > 2), S = \{a, a^{n+1}, c, cv\}. \]

If \(u \notin \langle a \rangle \text{ and } v \notin \langle c \rangle, \text{ then } G = \langle a \rangle \times \langle u \rangle \times \langle x \rangle = Z_n \times Z_2 \times Z_2(m > 2). \]

They are the cases (1)-(4) in Theorem 1.1, (ii), respectively.

Let \(\Gamma = Cay(G, S)\) be a directed Cayley graph for a finite group \(G\) on the set \(S\). Then \(\Gamma\) is normal edge-transitive if and only if \(Aut(G, S)\) is transitive on \(S\). It is known that if \(Aut(G, S)\) is transitive on \(S\), then all elements of \(S\) have the same order. Hence in Theorem 1.1, the cases 1, 3, 4 are not normal edge-transitive.
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