

A SUMMABILITY FACTOR THEOREM FOR ABSOLUTE SUMMABILITY INVOLVING QUASI POWER INCREASING SEQUENCES*

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Abstract – We obtain sufficient conditions for the series \( \sum a_n \lambda_n \) to be absolutely summable of order \( k \) by a triangular matrix.

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1. INTRODUCTION

Quite recently, Bor and Ozarslan [1] obtained sufficient conditions for \( \sum a_n \lambda_n \) to be summable \( \sum p_n, k \geq 1 \). Unfortunately, they used an incorrect definition of absolute summability (see, e.g. [2]). In this paper we obtain the corresponding result for triangular matrix using the correct definition of absolute summability. We obtain the correct form of [1] as a corollary.

Let \( T \) be a lower triangular matrix, \( \{ s_n \} \) a sequence. Then

\[ T_n = \sum_{v=0}^{n} t_{nv} s_v. \]

A series \( \sum a_n \) is said to be summable \( |T|_k, k \geq 1 \) if

\[ \sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{1} \]

We may associate with \( T \) two lower triangular matrices, \( \tilde{T} \) and \( \hat{T} \), defined as follows:

\[ \tilde{t}_{nv} = \sum_{r=0}^{n} t_{nr}, \quad n, v = 0, 1, 2, \ldots, \]

and

\[ \hat{t}_{nv} = \tilde{t}_{nv} - \tilde{t}_{n-1,v}, \quad n = 1, 2, 3, \ldots. \]

We may write

\[ T_n = \sum_{v=0}^{n} \tilde{t}_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=0}^{n} \tilde{t}_{nv} = \sum_{i=0}^{n} \tilde{a}_n a_i \lambda_i. \]

Thus

\[ T_n - T_{n-1} = \sum_{i=0}^{n} \tilde{a}_n a_i \lambda_i - \sum_{i=0}^{n-1} \tilde{a}_{n-1} a_i \lambda_i. \]

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\[= \sum_{i=0}^{n} \alpha_{in} a_i \lambda_i - \sum_{i=0}^{n} \alpha_{n-i,0} a_i \lambda_i\]

\[= \sum_{i=0}^{n} (\alpha_{ni} - \alpha_{n-i,0}) a_i \lambda_i\]

\[= \sum_{i=0}^{n} a_{ni} a_i \lambda_i = \sum_{i=1}^{n} \hat{a}_{ni} (s_i - s_{i-1})\]

\[= \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i s_i - \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i s_{i-1}\]

\[= \sum_{i=1}^{n-1} \hat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i s_{i-1}\]

\[= \sum_{i=1}^{n-1} \hat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_{i+1} s_i\]

\[= \sum_{i=1}^{n-1} (\hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1}) s_i + a_{nn} \lambda_n s_n\]

We may write

\[(\hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1}) = \hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1} - \hat{a}_{n,i+1} \lambda_{i+1} + \hat{a}_{n,i+1} \lambda_{i+1} = \lambda_{n,i+1} \Delta \hat{a}_{ni} + \hat{a}_{n,i+1} \Delta \lambda_i.\]

Therefore

\[T_n - T_{n-1} = \sum_{i=1}^{n-1} \Delta \hat{a}_{ni} \lambda_i s_i + \sum_{i=1}^{n-1} \hat{a}_{n,i+1} \Delta \lambda_i s_i + a_{nn} \lambda_n s_n.\]

\[= T_{n1} + T_{n2} + T_{n3}, \text{ say.}\]

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

A positive sequence \(\{\gamma_n\}\) is said to be a quasi \(\beta\) – power increasing sequence if there exists a constant \(K = K(\beta, \gamma) \geq 1\) such that \(K \gamma^n \geq m^n \gamma_m\) holds for all \(n \geq m \geq 1\).

It should be noted that every almost increasing sequence is a quasi \(\beta\) – power increasing sequence for any non-negative \(\beta\), but the converse need not be true as can be seen by taking the example, say \(\gamma_n = n^{-\beta}\) for \(\beta > 0\).

2. MAIN RESULT

Theorem 1. Let \(A\) be a lower triangular matrix with non-negative entries satisfying

(i) \(\overline{\alpha}_{n0} = 1, \ \ n = 0, 1,\ldots,\)

(ii) \(a_{n-1,v} \geq a_{nv}\) for \(n \geq v + 1\), and

(iii) \(na_{nn} = O(1)\).
Let \( \{X_n\} \) be a quasi \( \beta \)–power increasing sequence for some \( 0 < \beta < 1 \), and let \( \{\beta_n\} \) and \( \{\lambda_n\} \) be sequences such that

(i) \(|\Delta \lambda_n| \leq \beta_n\),

(ii) \(\lim \beta_n = 0\),

(iii) \(\sum_{n=1}^{\infty} n|\Delta \beta_n|X_n < \infty\), and

(iv) \(|\lambda_n|X_n = O(1)\)

if

(v) \(\sum_{n=1}^{\infty} \frac{1}{n} |s_n|^{\frac{1}{k}} = O(X_n)\),

then the series \(\sum a_n \lambda_n\) is summable \(|A|_k, k \geq 1\).

We need the following lemma for the proof of our theorem.

**Lemma ([3])**. Under the conditions on \( \{X_n\}, \{\beta_n\} \) and \( \{\lambda_n\} \) as taken in the statement of the theorem, the following conditions hold when (vi) is satisfied:

(a) \(n \beta_n X_n = O(1)\) and

(b) \(\sum_{n=1}^{\infty} \beta_n X_n < \infty\).

**Proof of Theorem 1**. The complete proof is sufficient, by Minkowski’s inequality, to show that

\[
\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty, \text{ for } r = 1, 2, 3.
\]

From the definition of \( \hat{A} \) and using (i) and (ii);

\[
\hat{a}_{n,i+1} = \pi_{n,i+1} - \pi_{n-1,i+1} = \sum_{v=i+1}^{n} a_{nv} - \sum_{v=i+1}^{n-1} a_{n-1,v} = 1 - \sum_{v=0}^{i} a_{nv} - \sum_{v=0}^{i} a_{n-1,v} = \sum_{v=0}^{i} (a_{n-1,v} - a_{nv}) \geq 0. \tag{2}
\]

From (vii), it follows that \( \lambda_n = O(1) \). Using Hölder’s inequality and (iii)

\[
I_1 := \sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} |\Delta \hat{a}_{ni}| |\lambda_i| |s_i|^k \right) \leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} |\Delta \hat{a}_{ni}| \right)^{k-1}. \]

**Summer 2006**

\[ \Delta_n \bar{a}_{ni} = a_{ni} - \bar{a}_{n,i+1} \]

\[ = \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1} \]

\[ = a_{ni} - a_{n-1,i} \leq 0. \]

Thus using (ii),

\[ \sum_{i=0}^{n-1} |\Delta_n \bar{a}_{ni}| = \sum_{i=0}^{n-1} (a_{n-1,i} - a_{ni}) = 1 - 1 + a_{mn} = a_{mn}. \] (3)

Using (iv), (vii), (viii) and the condition (b) of Lemma.

\[ I_1 : = O(1) \sum_{m=1}^{m+1} (n a_{mn})^{k-1} \sum_{i=1}^{n-1} |\Delta_n \bar{a}_{ni}| \lambda_i |s_i|^k \]

\[ : = O(1) \sum_{m=1}^{m+1} (n a_{mn})^{k-1} \sum_{i=1}^{n-1} |\Delta_n \bar{a}_{ni}| \lambda_i |s_i|^k \]

\[ : = O(1) \sum_{i=1}^{m} \lambda_i |s_i|^k \sum_{m=1}^{m+1} (n a_{mn})^{k-1} |\Delta_n \bar{a}_{ni}| \]

\[ : = O(1) \sum_{i=1}^{m} \lambda_i |s_i|^k a_{ni} \]

\[ : = O(1) \sum_{i=1}^{m} \lambda_i |s_i|^{k-1} \sum_{r=1}^{m-1} |s_r| |a_{ir} - \sum_{r=1}^{m} |s_r| a_{ir} \}

\[ : = O(1) \sum_{r=1}^{m-1} |s_r| |a_{ir} - \sum_{j=0}^{m-1} \lambda_{j+1} |s_r| a_{ir} \}

\[ : = O(1) \sum_{r=1}^{m-1} \Delta |s_r| |a_{ir} - \sum_{r=1}^{m} \lambda_{m} |s_r| a_{ir} \}

\[ : = O(1) \sum_{r=1}^{m-1} \beta_i X_i + |\lambda_{m} |X_m \]

\[ : = O(1). \]

Using Hölder’s inequality,

\[ I_2 \geq \sum_{n=1}^{m+1} n^{k-1} |T_{n2}| \leq \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=1}^{n-1} \bar{a}_{n,i+1} s_i \Delta \lambda_i \right|^k \]

\[ \leq \sum_{n=1}^{m+1} n^{k-1} ( \sum_{i=1}^{n-1} |s_i| |\Delta \lambda_i|^k )^k \]

\[ \leq \sum_{n=1}^{m+1} n^{k-1} ( \sum_{i=1}^{n-1} |s_i| \beta_i^k )^k \]
A summability factor theorem for…

\[
\sum_{n=1}^{m+1} n^{k-1}(\sum_{i=1}^{n-1} a_{n,i+1} |s_i|^k \delta_i) \leq (\sum_{i=1}^{m+1} a_{n,i+1} \delta_i)^{k-1}.
\]

It is easy to see that

\[
\sum_{i=1}^{n-1} a_{n,i+1} \delta_i \leq M a_m
\]

as in [4].

Using (iii)

\[
I_2 = O(1) \sum_{n=1}^{m+1} (na_m)^{k-1} \sum_{i=0}^{n-1} a_{n,i+1} |s_i|^k \delta_i
\]

\[
= O(1) \sum_{i=1}^{m} \beta_i |s_i|^k \sum_{n=1}^{m+1} (na_m)^{k-1} a_{n,i+1}
\]

\[
= O(1) \sum_{i=1}^{m} \beta_i |s_i|^k \sum_{n=1}^{m+1} a_{n,i+1}.
\]

From (2)

\[
\sum_{n=i+1}^{m+1} \left( \sum_{i=0}^{j} \left( a_{n-1,i} - a_{m,i} \right) \right) = \sum_{i=0}^{j} \sum_{n=1}^{m+1} \left( a_{n-1,i} - a_{m,i} \right)
\]

\[
= \sum_{i=0}^{j} \left( a_{m,i} - a_{m+1,i} \right)
\]

\[
\leq \sum_{i=0}^{j} a_{i,i} = 1.
\]

Using (v), (vi) and (viii)

\[
I_2 = O(1) \sum_{i=1}^{m} \beta_i |s_i|^k = O(1) \sum_{i=1}^{m} i \beta_i \frac{1}{i} |s_i|^k
\]

\[
= O(1) \sum_{i=1}^{m} i \beta_i \left[ \sum_{r=1}^{i} \frac{|s_r|^k}{r} - \sum_{r=1}^{i-1} \frac{|s_r|^k}{r} \right]
\]

\[
= O(1) \left[ \sum_{i=1}^{m} i \beta_i \left( \sum_{r=1}^{i} \frac{|s_r|^k}{r} - \sum_{j=1}^{m-1} (j+1) \beta_{j+1} \sum_{r=1}^{j} \frac{|s_r|^k}{r} \right) \right]
\]

\[
= O(1) \left[ \sum_{i=1}^{m} \Delta(i \beta_i) \sum_{r=1}^{i} \frac{|s_r|^k}{r} + O(1) m \beta_m \sum_{i=1}^{m} \frac{|s_i|^k}{i} \right]
\]

\[
= O(1) \left[ \sum_{i=1}^{m} \Delta(i \beta_i) |X_i| + O(1) m \beta_m X_m \right]
\]

\[
= O(1) \sum_{i=1}^{m-1} i |\Delta(\beta_i)| X_i + O(1) \sum_{i=1}^{m} \beta_{i+1} X_{i+1} + O(1) m \beta_m X_m
\]

\[
= O(1).
\]
again using the conditions of Lemma. Using (iii) and (vii),
\[
\sum_{n=1}^{m} n^{k-1} |T_{n,k}| \leq \sum_{n=1}^{m} n^{k-1} |a_{m,n} \lambda_n s_n|^k
\]
\[
= O(1) \sum_{n=1}^{m} (n a_{m,n})^{k-1} a_{m,n} |\lambda_n|^k |s_n|^k
\]
\[
= O(1) \sum_{n=1}^{m} a_{m,n} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k
\]
\[
= O(1),
\]
as in the proof of $I_1$.

**Corollary 1.** Let \( \{ p_n \} \) be a positive sequence such that \( P_n = \sum_{n=1}^{\infty} p_n \to \infty \), and satisfies (i) \( np_n = O(P_n) \).

Let \( \{ X_n \} \) be a quasi $\beta$ – power increasing sequence for some \( 0 < \beta < 1 \) and let \( \{ \beta_n \} \) and \( \{ \lambda_n \} \) be sequences such that

(i) \( |\Delta \lambda_n| \leq \beta_n \),

(ii) \( \lim \beta_n = 0 \),

(iii) \( \sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty \), and

(iv) \( \sum_{n=1}^{\infty} n |\Delta \lambda_n| X_n = O(1) \).

If

(vi) \( \sum_{n=1}^{m} \frac{1}{n} |s_n|^k = O(X_m) \),

then the series \( \sum a_{m,n} \lambda_n \) is summable \( |X, p_n| \), \( k \geq 1 \).

**Proof:** Conditions (ii), (iii), (iv), (v) and (vi) of Corollary 1 are, respectively, conditions (iv), (v), (vi), (vii) and (viii) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (iii) of Theorem 1 becomes condition (i) of Corollary 1.

**REFERENCES**