The Almost Sure Convergence for Weighted Sums of Linear Negatively Dependent Random Variables

H.R. Nili Sani,1,* M. Amini,2 and A. Bozorgnia2

1Department of Statistics, Faculty of Sciences, University of Birjand, Birjand, Islamic Republic of Iran
2Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran

Received: 27 April 2008 / Revised: 10 January 2009 / Accepted: 28 January 2009

Abstract

In this paper, we generalize a theorem of Shao [12] by assuming that \( \{X_n\} \) is a sequence of linear negatively dependent random variables. Also, we extend some theorems of Chao [6] and Thrum [14]. It is shown by an elementary method that for linear negatively dependent identically random variables with finite \( p \)-th absolute moment \( (p \geq 2) \) the weighted sums \( \frac{1}{A_n} \sum_{i=1}^{n} a_i X_i \) converge to zero as

\[
A_n = n^{1/p} \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2}
\]

where \( a_i \) is an array of real numbers. Moreover, we prove the almost sure convergence for weighted sums \( \sum_{i=1}^{n} a_{n,i} X_i \), \( n \geq 1 \), when \( \{X_i,i \geq 1\} \) is a sequence of pairwise negative quadrant dependence stochastically bounded random variables under some suitable conditions on \( a_{n,i} \).

Keywords: Pairwise negatively dependent; Linear negatively dependent; Negative association; Complete convergence

Introduction

Let \( \{X_n,n \geq 1\} \) be a sequence of independence random variables and suppose that \( \{a_{n,i},1 \leq i \leq n,n \geq 1\} \) is a double array of real numbers, the almost sure convergence of weighted sums \( \sum_{i=1}^{n} a_{n,i} X_i \) were studied by many authors (see, Chow [6], Thrum [14] and Sung [13]). Thrum [14] established the following extension of Chow [6].

Theorem 1. Suppose that \( \{X_n,n \geq 1\} \) is a sequence of i.i.d random variables with expectation zero and finite \( p \)-th \( (p \geq 2) \) absolute moment and \( \{a_{n,i},1 \leq i \leq n,n \geq 1\} \) is a sequence of nonrandom weighting coefficients with \( \sum_{i=1}^{n} a_{n,i}^2 = 1 \) for all \( n \geq 1 \). Then

\[
\sum_{i=1}^{n} a_{n,i} X_i / n^{1/p} \to 0 \text{ a.s. as } n \to \infty.
\]

But, in many stochastic models the assumption of

* Corresponding author, Tel.: +98(561)2502301, Fax: +98(561)2502041, E-mail: nilisani@yahoo.com
independence among random variables isn't plausible. In fact, increases in some random variables are often related to decreases in other random variables and the assumption of negative dependence is more appropriate than independence assumption. In the case of negative dependence and negative association dependence random variables some these results have been extended by other authors for example: Amini et al. [1], [2], Matula [10] and Nili Sani et al. [11] and Shao [12]. In this paper we generalize some results of Shao [12] and Thrum [14] for sequence \( \{X_n, n \geq 1\} \) of LIND random variables. Moreover we prove the almost sure convergence for weighted sums \( \sum_{i=1}^{n} a_n X_i \), when \( \{X_n, n \geq 1\} \) is a sequence of NQD stochastically bounded random variables under some suitable conditions on \( a_n \). In the following we present some definitions to be used in the proofs of our main results.

**Definition 1.** i) (Lehmann [8]). The random variables \( X \) and \( Y \) are said negatively quadrant dependent (NQD) if for each \( x, y \in \mathbb{R} \), \( P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y) \).

ii) The sequence \( \{X_i, i \geq 1\} \) of random variables is said to be pairwise NQD if \( \{X_i, Y_j\} \) is NQD for every \( i \neq j \).

iii) (Joag-Dev and Proschan [7]). The random variables \( X_1, \ldots, X_n \) are said to be negatively associated (NA) if for every pair of disjoint nonempty subsets \( A_1, A_2 \) of \( \{1, \ldots, n\} \),

\[
\text{Cov}(f_1(X_1, i \in A_1), f_2(X_1, i \in A_2)) \leq 0,
\]

whenever \( f_1 \) and \( f_2 \) are coordinatewise increasing and covariance exists.

**Definition 2.** A sequence \( \{X_n, n \geq 1\} \) of random variables is called asymptotically almost negatively associated (AANA) if there is a nonnegative sequence \( q(m) \to 0 \) such that

\[
\text{Cov}(f(X_n), g(X_{m+1}, \ldots, X_{m+k})) = q(m) \left( \text{Var}(f(X_n)) \text{Var}(g(X_{m+1}, \ldots, X_{m+k})) \right)^{1/2}
\]

for all \( m, k \geq 1 \) and for all coordinate increasing continuous functions \( f \) and \( g \) whenever the right side of (1) is finite.

**Definition 3.** The random variables \( X_1, X_2, \ldots, X_n \) are said to be linear negatively dependent (LIND) if for any disjoint \( A, B \subset \{1, \ldots, n\} \) and \( \lambda_j > 0, j = 1, \ldots, n \), \( \sum_{i \in A} \lambda_i X_i \) and \( \sum_{i \in B} \lambda_i X_i \) are NQD.

A sequence \( \{X_n, n \geq 1\} \) of random variables is said to be linear negatively dependent (pairwise NQD or negatively associated) if it holds for every finite subsequence.

Obviously, pairwise NQD sequence includes independent random variables and NA sequence, which has wide application in multivariate statistical analysis.

The following properties of the NA and NQD random variables and convex functions are based on our results.

(P1) Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated. (Joag-Dev and Proschan [7])

(P2) If \( \{X_n, n \geq 1\} \) is a sequence of pairwise NQD random variables, then

\[
\text{Cov}(X_i, X_j) \leq 0, \quad \forall \ i \neq j. \quad \text{(Bozorgnia et al. [3]).}
\]

(P3) If \( \{f_n\} \) is a sequence of Borel functions all monotone increasing (or all monotone decreasing) then \( \{f_n(X_n)\} \) is a sequence of pairwise NQD random variables (Bozorgnia et al. [3]).

(P4) Let \( (\Omega, F, P) \) be a probability space and \( \{A_n, n \geq 1\} \) is a sequence of pairwise NQD events. If \( \sum_{n=1}^\infty P(A_n) = \infty \), then \( P(\limsup_n A_n) = 1. \) (Matula [10]).

(P5) For any convex function \( f \) on \( \mathbb{R}^1 \), the right derivative \( f'_+ \) exists and is increasing. Moreover for all \( a, b, f(b) - f(a) = \int_a^b f'_+(t)dt \).

The next Theorem due to Shao [12]. We will extend this Theorem by assuming that \( \{X_n, n \geq 1\} \) is a sequence of LIND random variables.

**Theorem 2.** Let \( \{X_i, 1 \leq i \leq n\} \) be a negatively associated sequence, and let \( \{X_i^*, 1 \leq i \leq n\} \) be a sequence of independent random variables such that \( X_i^* \) and \( X_i \) have the same distribution for each \( i = 1, 2, \ldots, n \). Then

\[
E f\left( \sum_{i=1}^n X_i \right) \leq E f\left( \sum_{i=1}^n X_i^* \right)
\]

For any convex function \( f \) on \( \mathbb{R}^1 \), whenever the expectation on the right hand side of (2) exists.
Also, in this paper $c$ stands for a generic constant, not necessarily the same at each appearance.

**Results**

The next Lemma is an important technical tool in the proof of our main result. In fact in this Lemma we extend Theorem 2 for LIND random variables. Set

$$
\phi_m(q) = \sup_{\sigma} \sup_{\mu_1 B \mu_2} |P(B / A) - P(B)|,
$$

$$
\phi^* = \lim_{m \to \infty} \phi_m(q).
$$

**Lemma 1.** Let $\{a_i, i = 1, \ldots , j, j = 1, \ldots , n\}$ be a double array of non-negative real numbers and let $\{X_{i}, i = 1, \ldots , n\}$ be a sequence of non-negative LIND r.v.'s such that $X^{j} = \sum_{j=1}^{j} a_{ij}^{j} X_{j}$ and $\sum_{i=1}^{i} a_{i} a_{j}^{j} X_{i}^{j}$ are NQD. Assume that $\{X_{i}^{j}, i = 1, \ldots , n\}$ is a sequence of independent r.v.'s such that $X_{i}$ and $X_{i}^{j}$ have the same distribution for each $i = 1, \ldots , n$. Then

$$
E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j})) \leq E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j}))
$$

for any convex function $f$ on $R^1$, whenever the expectation on the right hand side of (3) exists.

**Proof.** By the same arguments of Shao [12], we prove (3), by induction on $n$. Let $(Y_{i}, Y_{j})$ be an independent copy of $(X_{i}, X_{j})$. It follows from (P5),

$$
E(f(cX_{i}X_{j})) + E(f(cY_{i}Y_{j}))
$$

$$
= E(f(cX_{i}Y_{j})) - E(f(cY_{i}X_{j}))
$$

$$
= E\left\{\frac{Y_{j}}{X_{j}} (cY_{j}f_{j}(ctY_{j}) - cX_{j}f_{j}(ctX_{j}))dt\right\}
$$

Since $f_{j}(x + t)$ is increasing functions of $x$ for each $t$, hence $f_{j}(X_{i} + t)$ and $I_{(X_{i} > \theta)}$ are NQD. By applying Fubini’s theorem, we conclude that

$$
2E(f(cX_{i}X_{j})) - E(f(cX_{i}X_{j}))
$$

$$
= E(f(cX_{i}X_{j})) + E(f(cY_{i}Y_{j}))
$$

$$
= E(f(cX_{j}Y_{j})) - E(f(cY_{j}X_{j}))
$$

$$
= E\left\{\int_{0}^{\tilde{Y}} (cY_{j}f_{j}(ctY_{j}) - cX_{j}f_{j}(ctX_{j}))dI_{(X_{j} > \theta)}\right\}
$$

by the induction hypothesis

$$
g(x) \leq E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j} + Z_{n-1}))
$$

$$
\leq E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j} + Z_{n-1}))
$$

$$
= E(g(X_{i} a_{i} X_{i}^{j}))
$$

$$
\leq E(f((X_{i} a_{i} X_{i}^{j} + \sum_{i=1}^{i} a_{i} X_{i}^{j})), by (4)
$$

$$
\leq E(f((X_{i} a_{i} X_{i}^{j} + \sum_{i=1}^{i} a_{i} X_{i}^{j})))
$$

$$
= E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j}))
$$

which proves (3) for $n = 2$. Let $Z_{n} = \sum_{i=1}^{i} a_{i} X_{i}^{j}$ and $g(x) = E(f(Z_{n-1} + x))$. By the induction hypothesis

$$
\phi^* = \lim_{m \to \infty} \phi_m(q)
$$

$$
\leq E(f((Z_{n-1} + Z_{n-1}))
$$

$$
= E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j} + Z_{n-1}))
$$

$$
= E(g(X_{i} a_{i} X_{i}^{j}))
$$

$$
\leq E(f((X_{i} a_{i} X_{i}^{j} + \sum_{i=1}^{i} a_{i} X_{i}^{j})), by (4)
$$

$$
\leq E(f((X_{i} a_{i} X_{i}^{j} + \sum_{i=1}^{i} a_{i} X_{i}^{j})))
$$

$$
= E(f(\sum_{i=1}^{i} a_{i} X_{i}^{j}))
$$

It is easy to show that $X_{i} a_{i} X_{i}^{j}, i = 1, \ldots , n - 1$, are LIND r.v.’s. Therefore, Theorem 2 completes the proof.

**Lemma 2.** Let $\{a_{i}, i = 1, 2, 3, \ldots \}$ be a double array of non-negative real numbers with $a_{i} = a_{j}$ for $i \neq j$ and $a_{i} = 0$ for all $i$. Assume that $\{X_{i}, i = 1, \ldots , n\}$ is a sequence of non-negative LIND random variables with
The next Lemma can be obtained from arguments of Thrum [14]. We omit the details.

**Lemma 3.** Suppose that $X_1, \cdots, X_s, \cdots$ are identically random variables with $E(X_j) = 0$ and $E(|X_j|^p) < \infty$ for some $p \in (0, 2)$ that satisfying in Marcinkiewicz and Zygmund’s theorem. Assume that nonrandom coefficients $b_{n,i}$ fulfill and $\sup \{|\sum b_{n,i}|, n \geq 1\} < \infty$.

Then

$$\sum_{j=1}^{s} b_{n,i} X_j \to N_{p}^{\infty} 0 \ a.s.$$

**Corollary 1.** Let $\{X_n, n \geq 1\}$ be a sequence of identically ANNA random variables with $\sum_{m=1}^{\infty} a_{n}^2 (m) < \infty$ (or pairwise NQD r.v.’s with $\phi'(1) < 1$). Assume that $E(X_1) = 0, E(|X_1|^p) < \infty$ for some $p \in (0, 2)$ and nonrandom coefficient $b_{n,i}$ fulfill $\sup \{|\sum b_{n,i}|, n \geq 1\} < \infty$. Then

$$\sum_{j=1}^{s} b_{n,i} X_j \to N_{p}^{\infty} 0 \ a.s.$$
\{a_n,1 \leq n, n \geq 1\} is an array of non-negative real numbers and \{X_i; n \geq 1\} is a sequence of non-negative random variables. The proof follows the same lines as the proof of Theorem 3 of Thrum [14]. It is clear that

\[ U_n^2 = \left( \sum_{i=1}^{n-1} a_i X_i / n^{1/2} \right)^2 = \sum_{i=1}^{n-1} a_i^2 X_i^2 / n^{2/2} \]

\[ \sum_{i=1}^{n-1} a_i a_j X_i X_j / n^{2/2} = V_n \cdot W_n. \]

\[ \lim_{n \to \infty} W_n = 0 \text{ a.s.} \]

Also, assume that \( n = 3 \) and \( a_i = 1, i \neq j \). It is easy to show that \( X^1 \sum_{j=1}^{n-1} a_{ij} X_j \) and \( \sum_{i \leq j, j \neq 1} a_{ij} X_j X_j \) are NQD.

In the following, we obtain the almost sure convergence for weighted sums \( \sum_{i} a_{n,i} X_i \), where \( \{X_i, n \geq 1\} \) is a sequence of pairwise NQD stochastically bounded random variables and \( \{a_n\} \) is an array of real numbers under some suitable conditions on \( a_{n,i} \).

**Theorem 4.** Let \( \{X_i, n \geq 1\} \) be a sequence of pairwise NQD random variables such that for all \( n \geq 1 \)

\[ P(|X_n| > x) \leq c \int_{x}^{\infty} e^{-\gamma t} dt \quad \gamma > 0; \]

let \( \{a_n, 1 \leq j \leq n\} \) be a triangular array of real numbers with \( \sum_{j=1}^{n} a_{n,j} = O(n^\beta), \quad \beta > 1. \)

Then

\[ \sum_{j=1}^{n} a_{n,j} X_j \xrightarrow{\text{a.s.}} 0 \]

**Proof:** By Cauchy Schwartz’s inequality we have

\[ \left| \sum_{j=1}^{n} a_{n,j} X_j \right|^2 \leq \left( \sum_{j=1}^{n} a_{n,j}^2 \right) \left( \sum_{j=1}^{n} X_j^2 \right) \leq c \cdot \left( \sum_{j=1}^{n} X_j^2 \right) . \]

Choosing an integer \( k \) such that \( 2^{k-1} \leq n < 2^k \), we get

\[ \limsup_{n \to \infty} \left( \sum_{j=1}^{n} a_{n,j} X_j \right)^2 \leq c \limsup_{n \to \infty} \frac{1}{(2^{k-1})^\beta} \left( \sum_{j=1}^{n} X_j^2 \right). \]

Therefore

\[ \sum_{k=1}^{\infty} P((2^{k-1})^\beta \left( \sum_{j=1}^{n} X_j^2 \right) > \varepsilon) \leq \frac{1}{\varepsilon} \sum_{k=1}^{\infty} (2^{k-1})^\beta \left( \sum_{j=1}^{n} X_j^2 \right) \leq c \sum_{k=1}^{\infty} (2^{(\beta)})^k < \infty \]

Now (P4) complete the proof.

**Corollary 3.** Under the assumptions of Theorem 4

i) If \( \beta > \alpha + 1 \), and for every \( \alpha > 0 \),

\[ \sum_{j=1}^{n} E(X_j^\alpha) = O(n^\alpha), \]

then

\[ \sum_{j=1}^{n} a_{n,j} X_j \xrightarrow{\text{a.s.}} 0 \]
ii) If \( a_{nj} = \frac{1}{n + j} \), \( 1 \leq j \leq n, n \geq 1 \) and \( \sum a_{nj}^3 = O(n^{2a-2}) \), then for every \( \alpha > 3/2 \),

\[
\sum_{i=1}^{n} a_{ni}X_i \xrightarrow{\text{a.s.}} 0.
\]

**Theorem 5.** Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise NQD random variables with \( EX_n = 0, n \geq 1 \), that are stochastically bounded by \( Z \sim N(0,1) \) and let \( \{a_{nj}, 1 \leq j \leq n, n \geq 1\} \) be a triangular array of real numbers such that

\[
\sum_{i=1}^{n} \left| a_{ni} - a_{n(i-1)} \right| = O(n^{-\beta}), \quad \beta > 1.
\]

Then

\[
\sum_{i=1}^{n} a_{ni}X_i \xrightarrow{\text{a.s.}} 0.
\]

**Proof.** By extension of Rademacher and Menschov’s inequality (Chandra and Chatterjee [4]) for any \( \epsilon > 0 \), we have

\[
\sum_{i=1}^{n} P(n^{-\beta} \max_{1 \leq i \leq n} |S_i| > \epsilon)
\]

\[
\leq \epsilon \sum_{i=1}^{n} n^{-2\beta} (\log n)^3 \sum_{j=1}^{n} EX_j^2
\]

\[
\leq \epsilon \sum_{i=1}^{n} n^{-2\beta-1}(\log n)^2 < \infty.
\]

Also, applying Abel’s summation rule we get

\[
\sum_{i=1}^{n} a_{ni}X_i \leq \max_{1 \leq i \leq n} \sum_{j=1}^{i} X_j \left| \sum_{i=1}^{n} \left( a_{ni} - a_{n(i-1)} \right) \right|
\]

\[
\leq cn^{-\beta} \max_{1 \leq i \leq n} \sum_{j=1}^{i} X_j
\]

Therefore

\[
\sum_{i=1}^{n} P(\sum_{j=1}^{n} a_{nj}X_j > \epsilon)
\]

\[
\leq \sum_{i=1}^{n} P(n^{-\beta} \max_{1 \leq j \leq n} |S_j| > \epsilon) < \infty.
\]

This and (P4) imply that

\[
\sum_{i=1}^{n} a_{ni}X_i \xrightarrow{\text{a.s.}} 0.
\]

**Acknowledgement**

The authors wish to thank the referees and the editor for their useful comments and suggestions.

**References**


